Unit 3 □ Vortex Motion

Structure

3.0 Introduction

It is well known that for irrotational motion the velocity vector $\mathbf{q} = (u,v,w)$ ca resented in the form of the gradient of a velocity potential ϕ as

$$\mathbf{q} = \text{grad}\ \phi$$

or, in other words,

$$u = \frac{\partial \phi}{\partial x},\ v = \frac{\partial \phi}{\partial y},\ w = \frac{\partial \phi}{\partial z}.$$

The *vorticity* is defined to be a vector $\Omega = \text{curl}\ \mathbf{q}$, whose components are

$$\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z},\ \Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},\ \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

The above components vanish when the conditions (1) are satisfied. Thus tational motion when $\mathbf{q} = \text{grad}\phi$,

$$\Omega = \text{curl grad}\ \phi = 0.$$

Conversely, if $\Omega = 0$, then with the aid of vector analysis, it can be shown that equation (1) will always hold. Thus, in irrotational motion, a velocity potential certainly exists.

This chapter will consist of investigation of such motions of a fluid for which the *vorticity vector* Ω is different from zero at least in some part of the fluid under consideration. We will call such motions as vortex motions of the fluid.

3.1 Vortex lines and Vortex tubes

A vortex line is a curve in the fluid such that its tangent at any point gives the direction of the local vorticity. Therefore, the equations of a vortex line have the form

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z} \tag{4}$$

where Ω_x, Ω_y, Ω_z are the components of the vorticity vector Ω. Note that, the above equations are analogous to the equations for a streamlines. Portions of the fluid bounded by vortex lines through every point of an infinitely small closed curves are called vortex filaments, or simply vortices. Vortex lines passing through any closed curve form a tubular surface, which is called a *vortex tube*. The fluid contained within such a tube constitutes what is called a vortex-filament. Let δS_1, δS_2 be two sections of a vortex tube and let $\mathbf{n_1}$ and $\mathbf{n_2}$ be the unit normals to these sections drawn outwards from the fluid between them. Also, let δS be the curved surface of the vortex tube. Then, $\Delta S = \delta S_1 + \delta S_2 + \delta S$ = total surface area of the element. Let ΔV be the total volume contained in ΔS. Then

$$\int_{\Delta S} \mathbf{n}.\Omega dS = \int_{\Delta V} \text{div}\Omega dV = 0,$$

since div $\Omega = 0$. Thus

$$\int_{\delta S_1} \mathbf{n}.\Omega dS = \int_{\delta S} \mathbf{n}.\Omega dS + \int_{\delta S_2} \mathbf{n}.\Omega dS = 0.$$

At each point of δS, $\mathbf{n}.\Omega = 0$, since Ω is tangential to the curved surface. Thus

$$(\mathbf{n_1}.\Omega)\delta S_1 + (\mathbf{n_2}.\Omega)\delta S_2 = 0$$

approximately to the first order (using the mean value theorem of integral calculus). This shows that $|\mathbf{n}.\Omega| dS$ is constant for every section δS of the vortex tube. Its value is called the *strength* of the vortex tube. A vortex tube whose strength is unity is called a *unit vortex tube*.

Some ...

(1) Vortex lines and tubes move with the fluid.

Let C be any closed curve drawn on the surface of the vortex tube containing an area S of the tube and not embracing the tube. As the vortieity vectors are everywhere lying on the surface S, it follows that, $\mathbf{n}.\Omega = 0$. So the circulation Γ around C is given by

$$\int_\Gamma \mathbf{q}\cdot d\mathbf{s} = \int_S \mathbf{n}.\Omega dS = 0.$$

After an interval of time, the same fluid particles form a new surface, say S′. According to Kelvin's theorem, the circulation around S′ must also be zero. As this is true for any S, the component of vortieity normal to every element of S′ must vanish, showing that S′ must lie on the surface of the vortex tube. Hence, vortex lines and vortex tubes move with fluid.

(2) Vortex lines and tubes move with the fluid.

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(3) A vortex tube cannot originate or end within the fluid. It must either end at a solid boundary or form a closed loop (a 'vortex ring').

Suppose S is any closed surface containing a volume V. Then

$$\int_S \mathbf{n}.\Omega dS = \int_V \operatorname{div}\Omega dV = 0. \tag{5}$$

Equation (5) shows that the total strength of vortex tubes emerging from S is equal to that entering S. This means that *vortex lines and tubes cannot originate* or *terminate at internal points in a fluid.* They can only form closed curves or terminate on boundaries.

(4) Strength of a vortex tube remains constant for all time.

If C is a closed curve embracing once the vortex tube and if S denotes an area contained in C, then the circulation Γ of the fluid velocity $\mathbf{q}$ around the vortex tube is defined as

$$\Gamma = \oint_C \mathbf{q}.ds \tag{6}$$

Then, by Stokes' theorem

$$\Gamma = \int_S \mathbf{n}.qdS. \tag{7}$$

Equation (7) shows that Γ is nothing but the strength of vortex tube with surface area S. Since for an inviseid fluid the circulation around any closed curve in the fluid moving along with the fluid, remains constant in time, therefore strength of the vortex also remains constant in time.

The above theorems are known as **Helmholtz's vortex theorems :**

We shall assume that the fluid is a single-valued function of time only.

3.2 Rectilinear Vortex

Consider a single tube whose cross-section is a circle of radius a and with its axis parallel to the axis of z surrounded by unbounded fluid. The motion is similar in all planes parallel to xy and it has no velocity along the axis of z. By making the area contained within the tube sufficiently small we see that the distribution producing such a flow must be uniform along the z-axis. Such a distribution along the z-axis is called a uniform *rectilinear* or *line vortex*. Thus if $\mathbf{q} = (u, v, w)$ be the velocity, then $w = 0$ and u, v are independent of z. If $\Omega = (\Omega_x, \Omega_y, \Omega_z)$ be the vorticity vector, then

$$\Omega_x = 0, \Omega_y = 0, \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{8}$$

The velocity components u, v are related to the stream function ψ by

$$u = -\frac{\partial \psi}{\partial y} \text{ and } v = \frac{\partial \psi}{\partial x}. \tag{9}$$

Use of (9) in (8) gives

$$\Omega_z = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}. \tag{10}$$

Thus, ψ satisfies

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \begin{cases} \Omega_x, & \text{on the vortex,} \\ 0, & \text{out side the vortex.} \end{cases} \tag{11}$$

Let P(r, θ) be any point outside the vortex. Since the motion outside the vortex is irrotational, the velocity potential ϕ exists and

$$\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \tag{12}$$

holds, r, θ being polar coordinates. Since, in the region out side the vortex ψ is harmonic so we get

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0. \tag{13}$$

If the motion is symmetric about the origin, ψ must be independent of θ. Then equation (13) reduces to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = 0$$

giving

$$\psi = c \log r, \; c = \text{constant}. \tag{14}$$

Using the relation of ϕ and ψ given by (12) we get

$$\phi = -c\theta. \tag{15}$$

Thus the complex potential function w is given by

$$w = \phi + i\psi = -c\theta + ic \log r = ic \log z. \tag{16}$$

Let k be the circulation in the circuit enclosing the vortex. Then

$$k = \int_0^{2\pi}\left(-\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) r d\theta = 2\pi c$$

so that

$$c = \frac{k}{2\pi}$$

and hence w is given by

$$w = \frac{ik}{2\pi} \log z. \tag{17}$$

This is the complex potential due to a vortex of strength k placed at the origin. If the vortex be placed at $z_0 = x_0 + iy_0$, instead of (0, 0), then the complex potential w has the form

$$w = \frac{ik}{2\pi} \log(z - z_0). \tag{18}$$

Let $P(z) \equiv P(x, y)$ be another point in the fluid other than (x_0, y_0). Then distance r_0 between (x, y) and (x_0, y_0) is given by

$$r_0^2 = (x - x_0)^2 + (y - y_0)^2. \tag{19}$$

From (18), we see that the stream function ψ is given by

$$\psi = \frac{k}{2\pi} \log r_0.$$

Thus,

$$u = -\frac{\partial \psi}{\partial y} = -\frac{\partial \psi}{\partial r_0}\frac{\partial r_0}{\partial y} = -\frac{k}{2\pi} \cdot \frac{y - y_0}{r_0^2}$$

and

$$v = \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial r_0}\frac{\partial r_0}{\partial x} = \frac{k}{2\pi} \cdot \frac{x - x_0}{r_0^2}.$$

Thus the magnitude of the velocity **q** is given by

$$q = \left(u^2 + v^2\right)^{\frac{1}{2}} = \frac{k}{2\pi r_0}.$$

This is the velocity at any point P(x, y) due to presence of a vortex of strength k at (x_0, y_0).

Note :

If there be any number of vertices of strength k_s at z_s, s = 1, 2, 3, ..., then the complex potential at any point z in the fluid is given by

$$w = \frac{i}{2\pi}\sum_s k_s \log(z - z_s),$$

and the velocity components are given by

$$u = -\frac{1}{2\pi}\sum_s k_s \frac{(y - y_s)}{r_s^2} \text{ and } v = \frac{1}{2\pi}\sum_s k_s \frac{(x - x_s)}{r_s^2}$$

where

$$z_s = x_s + iy_s \text{ and } r_s^2 = (x - x_s)^2 + (y - y_s)^2.$$

Let (u_s, v_s) denote the velocity components of the vortex of strength k_s. Then

$$u_s = -\frac{1}{2\pi}\sum_{r \neq s} k_r \frac{(y_r - y_s)}{R_{r_s}^2} \text{ and } v_s = \frac{1}{2\pi}\sum_{r \neq s} k_r \frac{(x_r - x_s)}{R_{r_s}^2}$$

where

$$R_{r_s}^2 = (x_r - x_s)^2 + (y_r - y_s)^2.$$

Note that the expressions $\sum k_s u_s$ and $\sum k_s v_s$ will consist of pairs of terms of the forms

$$k_r \cdot \frac{k_s}{2\pi} \frac{(x_r - x_s)}{R_{r_s}^2} \text{ and } k_s \cdot \frac{k_r}{2\pi} \frac{(x_s - x_r)}{R_{r_s}^2}$$

and as such

$$\sum k_s u_s = 0 \text{ and } \sum k_s v_s = 0.$$

Hence, regarding k as a mass, the center of gravity of the vortex system, viz.

$$\bar{x} = \frac{\sum k_s u_s}{\sum k_s}, \quad \bar{y} = \frac{\sum k_s v_s}{\sum k_s}$$

remains stationary throughout the motion. Note that if $\sum k_s = 0$, the center $(\bar{x}, \bar{y})$ is at infinity.

3.3 Circular Vortex

Let there be a single cylindrical vortex tube, whose cross-section is a circle of radius a, surrounded by unbounded fluid.

The section of the vortex by the plane of the motion is a circle and the arrangement may therefore be referred to as a *circular vortex*.

3.3.1 Vortex pair

Consider the case of two vortices of strengths k_1 and k_2 at a distance r_0 apart. Let A, B be their centers, O, the center of the system. The point O divides AB in the ratio $k_2 : k_1$. The motion of each vortex as a whole is entirely due to the other, and is therefore always perpendicular to AB. Hence the two vortices remain ahvays at the same distance from one another and rotate with constant angular velocity about O which is fixed. The velocities at the two vortices at A and B are respectively $\frac{k_1}{2\pi r_0}$ and $\frac{k_2}{2\pi r_0'}$. To obtain the angular velocity w of the system, we divide the velocity of the vortex A by the distance AO, where

$$AO = \frac{k_2}{k_1 + k_2} . AB = \frac{k_2 r_0}{k_1 + k_2}.$$

Therefore, the angular velocity is given by

$$\omega = \frac{\text{velocity of the vortex at A}}{AO} = \frac{k_1 + k_2}{2\pi r_0^2}.$$

If k_1, k_2 be of the same sign, i.e. if the direction of rotation in the two vortices be the same then O lies between A and B; otherwise O lies in AB or BA, produced. If $k_1 = -k_2$, O is at infinity. However, A, B move with equal velocities $\frac{k_1}{2\pi r_0}$ at right angles to AB, which remains fixed in direction. Such a combination of two equal and opposite vortices may be called a *vortex pair*.

3.3.2 Vortex doublet

Consider a vorlex pair, k at $ae^{i\alpha}$ and $-k$ at $-ae^{i\alpha}$ in the complex z-plane where $z = x + iy$. If we let $a \to 0$ and $k \to \infty$ so that $2ak = \mu$ is a finite constant, we get a vortex doublet of strength μ inclined at an angle α to the x-axis.

The direction of the doublet is determined from the vortex of negative rotation to that of positive rotation. The complex potential is

$$w = \lim_{a \to 0} \frac{ik}{2\pi} \{\log(z - ae^{i\alpha}) - \log(z + ae^{i\alpha})\}$$

$$= \lim_{a \to 0} \frac{ik}{2\pi} \left(-\frac{ae^{i\alpha}}{z} + \frac{a^2 e^{2i\alpha}}{2z^2} - \dots - \frac{ae^{i\alpha}}{z} - \frac{a^2 e^{2i\alpha}}{2z^2} - \dots \right) = -\frac{i\mu}{2\pi z} e^{2i\alpha},$$

The stream function is $\psi = -\dfrac{\mu}{2\pi r}\cos(\alpha - \theta)$.

If, in particular, we take the vortex doublet to be at the origin and along the axis of y, we have $\psi = -\dfrac{\mu \sin\theta}{2\pi r}$. If we put $\dfrac{\mu}{2\pi} = Ub^2$, we obtain $\psi = -\dfrac{Ub^2 \sin\theta}{r}$ which is the stream function for a circular cylinder of radius b moving with velocity U along the x-axis.

Thus the motion due to a circular cylinder is the snme as that due to a suitable vortex doublet placed at the center, and with its axis perpendicular to the direction of motion.

3.4 Infinite row of parallel rectilinear vortices

3.4.1 Single infinite row

Consider an infinite row of vortices each of strength k at the points 0, ±a, ±2a, ... ±na, ... (as shown in figure 3.1).

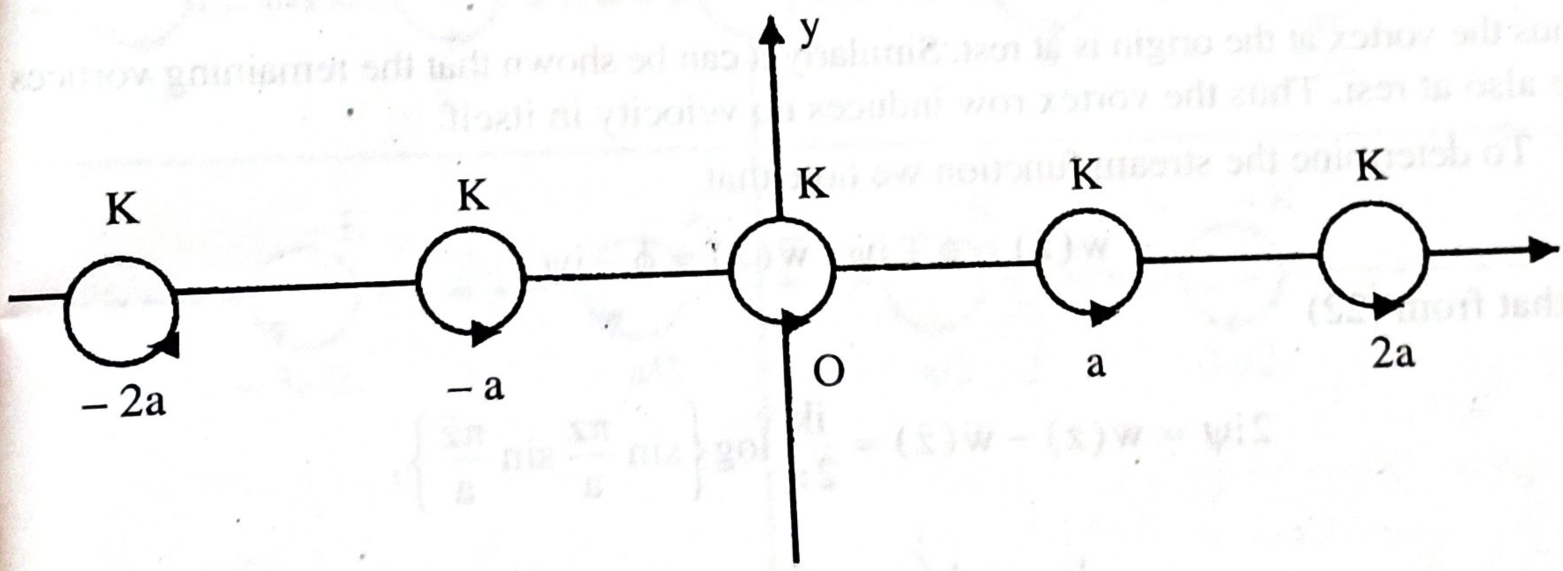

Figure 3.1

The complex potential of the (2n + 1) vortices nearest to the origin is

$$w_n = \frac{ik}{2\pi}\log z + \frac{ik}{2\pi}\log(z-a) + \cdots + \frac{ik}{2\pi}\log(z-na)$$

$$+\frac{ik}{2\pi}\log(z+a) + \cdots + \frac{ik}{2\pi}\log(z+na)$$

$$= \frac{ik}{2\pi}\log\{z(z^2-a^2)(z^2-2^2a^2)\cdots(z^2-n^2a^2)\}$$

$$= \frac{ik}{2\pi}\log\left\{\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\cdots\left(1-\frac{z^2}{n^2a^2}\right)\right\} + \frac{ik}{2\pi}\log\left\{\frac{a}{\pi}.a^2.2^2a^2\cdots n^2a^2\right\}$$

The constant term may be omitted, so that we write

$$w_n = \frac{ik}{2\pi}\log\left\{\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\cdots\left(1-\frac{z^2}{n^2a^2}\right)\right\}. \qquad (20)$$

Now, sin x can be expressed as an infinite product in the form

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\cdots\left(1-\frac{x^2}{n^2\pi^2}\right)\cdots \qquad (21)$$

Thus letting $n \to \infty$ in (20), we get by virtue of (21),

$$w = \frac{ik}{2\pi}\log\sin\left(\frac{\pi z}{a}\right). \qquad (22)$$

Consider the vortex at z = 0. Since its moton is due to the other vortices, the complex velocity of the vortex at the origin is given by

$$-\frac{d}{dz}\left\{\frac{ik}{2\pi}\log\sin\frac{\pi z}{a} - \frac{ik}{2\pi}\log z\right\}_{z=0} = -\frac{ik}{2\pi}\left(\frac{\pi}{a}\cdot\cot\frac{\pi z}{a} - \frac{1}{z}\right)_{z=0} = 0.$$

Thus the vortex at the origin is at rest. Similarly it can be shown that the remaining vortices are also at rest. Thus the vortex row induces no velocity in itself.

To determine the stream function we note that

$$w(z) = \phi + i\psi,\ \overline{w}(\overline{z}) = \phi - i\psi$$

so that from (22)

$$2i\psi = w(z) - \overline{w}(\overline{z}) = \frac{ik}{2\pi}\log\left\{\sin\frac{\pi z}{a}\sin\frac{\pi\overline{z}}{a}\right\},$$

$$\psi = \frac{k}{4\pi}\log\frac{1}{2}\left(\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}\right).$$

For large values of $\frac{y}{a}$, we neglect the term $\cos\frac{2\pi x}{a}$, for its modulus never exceeds unity, and therefore along the streamlines ψ = constant. Thus at a great distance from the row the stream lines are parallel to the row.

Again, if v_1, v_2 are the complex velocities at the points z, $\bar{z}$ respectively, we have

$$v_1 + v_2 = -\frac{d}{dz}\left\{\frac{ik}{2\pi}\log\sin\frac{\pi z}{a}\right\}_{z=z} - \frac{d}{dz}\left\{\frac{ik}{2\pi}\log\sin\frac{\pi z}{a}\right\}_{z=\bar{z}}$$

$$= -\frac{ik}{2a}\cot\frac{\pi z}{a} - \frac{ik}{2a}\cot\frac{\pi\bar{z}}{a} = -\frac{ik}{2a}\frac{2\sin\frac{2\pi x}{a}}{\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}},$$

which is purely imaginary and tends to zero when y tends to infinity. Thus the velocities along the distant streamlines are parallel to the row but in opposite directions.

3.4.2 Infinite row of parallel rectilinear vortices (Karman Vortex Street)

This consists of two parallel infinite rows of the same spacing, say a, but of opposite vortex strengths k and – k, so arranged that each vortex of the upper row is directly above

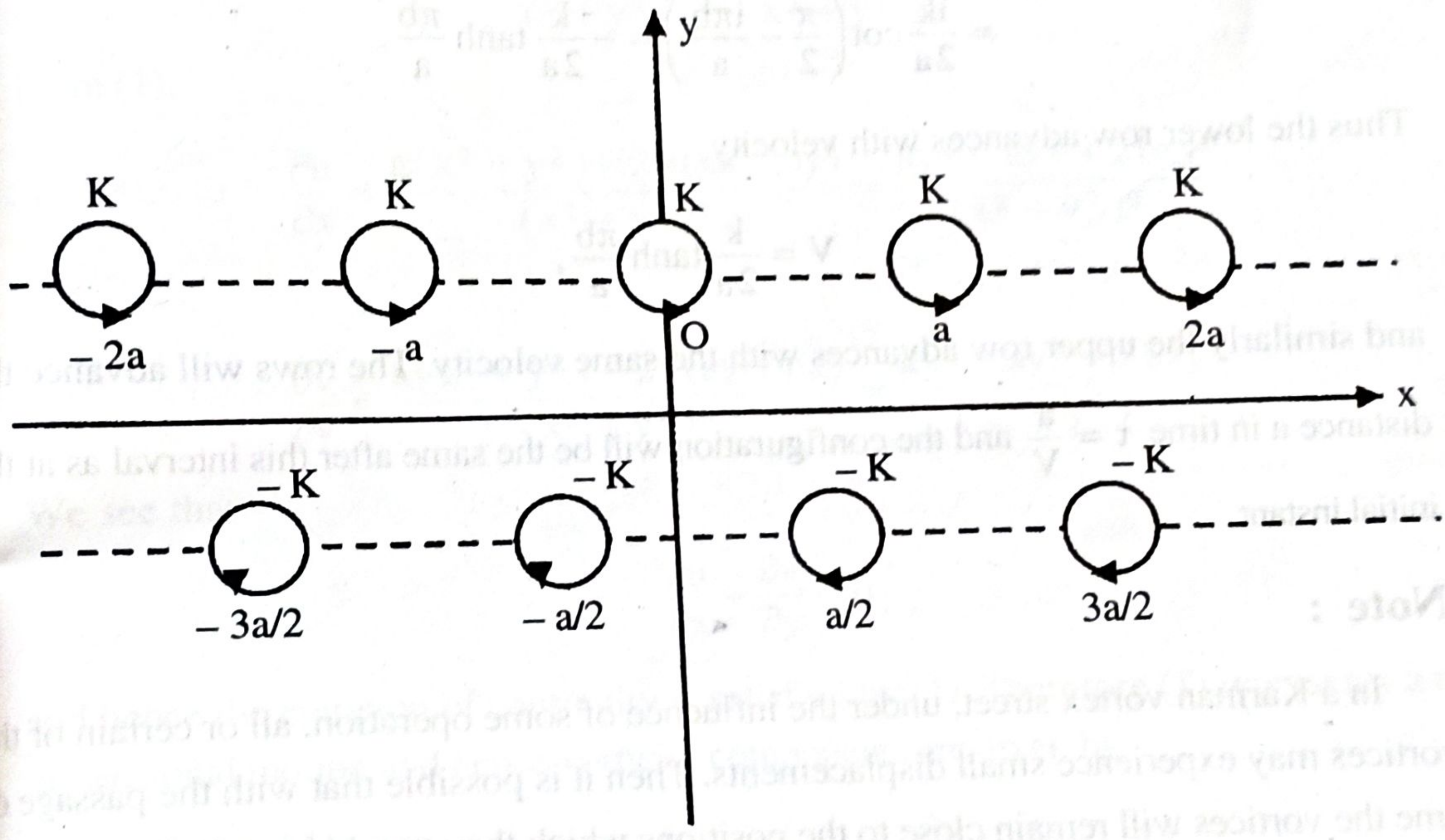

Figure 3.2

the mid point of the line joining two vortices of the lower row and vice-versa. Taking the configuration at time $t = 0$, we take the axes as shown in the figure 3.2, the x-axis being midway between and parallel to the rows which are at the distance b apart. At this instant the vortices in the upper row are at the points $ma + \frac{1}{2}ib$, and those in the lower row at the points $\left(m+\frac{1}{2}\right)a - \frac{1}{2}ib$, where $m = 0, \pm1, \pm2, \ldots$

The complex potential at the instant $t = 0$, by the preceding section is given by

$$w = \frac{ik}{2\pi}\log\sin\frac{\pi}{a}\left(z - \frac{ib}{2}\right) + \frac{ik}{2\pi}\log\sin\frac{\pi}{a}\left(z - \frac{a}{2} + \frac{ib}{2}\right).$$

Since neither row induces any velocity in itself, the velocity of vortex at $z = \frac{a}{2} - \frac{ib}{2}$ will be given by

$$-u_1 + iv_1 = \left[\frac{d}{dz}\frac{ik}{2\pi}\sin\frac{\pi}{a}\left(z - \frac{ib}{2}\right)\right]_{z=\frac{a}{2}-\frac{ib}{2}}$$

$$= \frac{ik}{2a}\cot\left(\frac{\pi}{2} - \frac{i\pi b}{a}\right) = -\frac{k}{2a}\tanh\frac{\pi b}{a}.$$

Thus the lower row advances with velocity

$$V = \frac{k}{2a}\tanh\frac{\pi b}{a},$$

and similarly the upper row advances with the same velocity. The rows will advance the distance a in time $\tau = \frac{a}{V}$ and the configuration will be the same after this interval as at the initial instant.

Note :

In a Karman vortex street, under the influence of some operation, all or certain of the vortices may experience small displacements. Then it is possible that with the passage of time the vortices will remain close to the positions which they would have had if they had not been subject to displacements. We then say that the motion is stable. If, however, t displaced vortices tend to move away from the position corresponding to unperturbed state, the motion will be called unstable. A necessary condition of stability for the Karman's vortex street is

$$\cosh\frac{b\pi}{a} = \sqrt{2}$$

so that b = 0.281a.

3.5 Illustrative Solved Examples

Example 1

If

$$u = \frac{ax - by}{x^2 + y^2},\ v = \frac{ay + bx}{x^2 + y^2},\ w = 0,$$

investigate the nature of motion of the liquid.

Solution :

Given

$$u = \frac{ax - by}{x^2 + y^2},\ v = \frac{ay + bx}{x^2 + y^2},\ w = 0. \tag{1}$$

From (1),

$$\frac{\partial u}{\partial x} = \frac{a(x^2 + y^2) - 2x(ax - by)}{(x^2 + y^2)^2} = \frac{ay^2 - ax^2 + 2bxy}{(x^2 + y^2)^2}$$

and

$$\frac{\partial v}{\partial y} = \frac{a(x^2 + y^2) - 2y(ay + bx)}{(x^2 + y^2)^2} = \frac{ax^2 - ay^2 - 2bxy}{(x^2 + y^2)^2}.$$

We see that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

and hence the equation of continuity is satisfied by (1). Therefore (1) represents a two-dimensional motion and hence vorticity components are given by

$$\Omega_x = 0,\ \Omega_y = 0,\ \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \quad (2$$

From (1),

$$\frac{\partial u}{\partial y} = \frac{-b(x^2+y^2) - 2y(ax-by)}{(x^2+y^2)^2} = \frac{by^2 - bx^2 - 2axy}{(x^2+y^2)^2}$$

and

$$\frac{\partial v}{\partial x} = \frac{b(x^2+y^2) - 2x(ay+bx)}{(x^2+y^2)^2} = \frac{by^2 - bx^2 - 2axy}{(x^2+y^2)^2}$$

so that $\Omega_z = 0$. Thus

$$\Omega_x = 0, \Omega_y = 0, \Omega_z = 0$$

showing that the motion is irrotational.

Example 2

Find the necessary and sufficient conditions that vortex lines may be at right angles the streamlines.

Solution :

Streamlines and vortex lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

and

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z}$$

respectively. These will be at right angles, if

$$u\Omega_x = v\Omega_y = w\Omega_z.$$

But

$$\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

Using (4), (3) may be written as

$$u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0,$$

which is the necessary and sufficient condition that udx + vdy + wdz may be a perfe differential. So we may write

$$udx + vdy + wdz = \mu d\phi = \mu\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right).$$

Thus the necessary and sufficient conditions that vortex lines may be at right angles to the streamlines are

$$u = \mu\frac{\partial\phi}{\partial x},\ v = \mu\frac{\partial\phi}{\partial y}, w = \mu\frac{\partial\phi}{\partial z}.$$

Example 3

When an infinite liquid contains two parallel, equal and opposite rectilinear vortices at a distance 2b, prove that the streamlines relative to this system are given by the equation

$$\log\frac{x^2 + (y-b)^2}{x^2 + (y+b)^2} + \frac{y}{b} = C,$$

the origin being the midpoint of the line joining the two vortices, taken as the y-axis.

Solution :

Let there be two rectilinear vortices of strengths k and – k at $P_1(0, b)$ and $P_2(0, -b)$ respectively. Thus $P_1P_2 = 2b$, origin being the midpoint of P_1P_2 and y-axis being taken along P_1P_2. Thus we have a vortex pair which will move with a uniform velocity $k/2\pi P_1P_2$ or $k/4\pi b$ perpendicular to the line P_1P_2 (ie. along the x-axis). To determine the streamlines relative to the vortices, we must impose a velocity on the given system equal and opposite to the velocity $k/4\pi b$ of motion of the vortex pair. Accordingly, we add a term $\frac{kz}{4\pi b}$ to the complex potential of the vortex pair. Note that

$$-\frac{d}{dz}\left(\frac{kz}{4\pi b}\right),$$

and hence the term added is justified. So, for the case under consideration, the complex potential is given by

$$w = \phi + i\psi = \frac{ik}{2\pi}\log(z - ib) - \frac{ik}{2\pi}\log(z + ib) + \frac{kz}{4\pi b}.$$

Equating the imaginary parts, we have

$$\psi = \frac{k}{4\pi}\log[x^2 + (y-b)^2] - \frac{k}{4\pi}\log[x^2 + (y+b)^2] + \frac{kz}{4\pi b}$$

$$\psi = \frac{k}{4\pi}\left[\log\frac{x^2+(y-b)^2}{x^2+(y+b)^2}+\frac{y}{b}\right].$$

Hence the required relative streamlines are given by ψ = constant, i.e.,

$$\log\frac{x^2+(y-b)^2}{x^2+(y+b)^2}+\frac{y}{b}=C.$$

Example 4

If n rectilinear vortices of the same strength k are symmetrically arranged as generators of a circular cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $8\pi^2a^2/(n-1)k$, and find the velocity of any part, of the liquid.

Solution :

Let us take the origin as the center of the circle of radius a and the x-axis along the line $\theta = 0$. Suppose that n rectilinear vortices each of strength k be situated at points $z_m = a\exp^{2\pi im/n}$, m = 0, 1, 2, ..., n – 1 on the circumference of the circle. Then the complex potential due to these n vortices is given by

$$w = \frac{ik}{2\pi}\sum_{m=0}^{n-1}\log(z-a\exp^{2\pi im/n})$$

$$=\frac{ik}{2\pi}\prod_{m=0}^{n-1}(z-a\exp^{2\pi im/n})=\frac{ik}{2\pi}\log(z^n-a^n).$$

Now the fluid velocity q at any point out of all the n vortices is given by

$$q=\left|-\frac{dw}{dz}\right|=\left|\frac{ik}{2\pi}\frac{z^{n-1}}{z^n-a^n}\right|=\left|\frac{kn}{2\pi}\frac{z^n-1}{z^n-a^n}\right|.$$

Again the velocity induced at the point z = a, by the other vortices is given by the complex potential

$$w' = \frac{ik}{2\pi}\log(z^n-a^n)-\frac{ik}{2\pi}\log(z-a)$$

so that

$$w - \frac{dw}{dz} = \frac{ik}{2\pi} \log\left(z^{n-1} + z^{n-1} a + \cdots + z a^{n-1} + a^{n-1}\right).$$

Hence

$$\left(\frac{dw'}{dz}\right)_{z=a} = \frac{ik}{2\pi} \frac{(n-1)+(n-2)+\cdots+2+1}{na} = \frac{ik(n-1)}{4\pi a}$$

or

$$u_1 - iv_1 = \left(\frac{dw'}{dz}\right)_{z=a} = -\frac{ik(n-1)}{4\pi a}$$

so that $u_1 = 0$ and $v_1 = \frac{k(n-1)}{4\pi a}$. If q_r and q_θ be the radial and transverse velocity components of the velocity at $z = a$, then we have $q_r = 0$ and $q_\theta = \frac{k(n-1)}{4\pi a}$. Due to symmetry of the problem, it follows that each vortex moves with the same transverse velocity $\frac{k(n-1)}{4\pi a}$. Hence the required time T is given by

$$T = \frac{2a\pi}{\frac{k(n-1)}{4\pi a}} = \frac{8\pi^2 a^2}{(n-1)k}.$$