Vector Integration (I)

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Vector Integration :

If a vector function $R(u)$ of a scalar variable u be such that $a(u) = \frac{d}{du} R(u)$ then $\int a(u)\, du = R(u) + c$, c being arbitrary constant vector independent of u.

$R(u)+c$ is called the indefinite integral of $a(u)$.

Let $a(u) = a_1(u)\hat{i} + a_2(u)\hat{j} + a_3(u)\hat{k}$

where $a_1(u)$, $a_2(u)$, $a_3(u)$ are supposed to be continuous in a specified interval.

$$\therefore \int a(u)\, du = \hat{i}\int a_1(u)\, du + \hat{j}\int a_2(u)\, du + \hat{k}\int a_3(u)\, du .$$

A definite integral of $a(u)$ between the limits $u=a$ and $u=b$ can be written as

$$\int_a^b a(u)\, du = \int_a^b \frac{d}{du} R(u)\, du$$

$$= \left[R(u)+c\right]_a^b$$

$$= R(b) - R(a) .$$

Some results :

(i) $\int \left(\vec{r}\cdot \frac{d\vec{A}}{dt} + A\cdot \frac{d\vec{r}}{dt}\right) dt = \int \frac{d}{dt}\left(\vec{r}\cdot\vec{A}\right) dt = \vec{r}\cdot\vec{A} + c$.

(ii) $\int \left(2\vec{r}\cdot \frac{d\vec{r}}{dt}\right) dt = \int \frac{d}{dt}\left(\vec{r}^{\,2}\right) dt = \vec{r}^{\,2} + c$.

(iii) $\int \left(2\, \frac{d\vec{r}}{dt}\cdot \frac{d^2\vec{r}}{dt^2}\right) dt = \int \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)^2 dt = \left(\frac{d\vec{r}}{dt}\right)^2 + c$.

(iv) $\int \left(\frac{d\vec{r}}{dt}\times\vec{s} + \vec{r}\times\frac{d\vec{s}}{dt}\right) dt = \int \frac{d}{dt}\left(\vec{r}\times\vec{s}\right) dt = \vec{r}\times\vec{s} + c$

(v) $\int \left(\vec{a}\times\frac{d\vec{r}}{dt}\right) dt = \vec{a}\times\vec{r} + c$, $\vec{a}$ being constant vector.

(vi) $\int \left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right) dt = \vec{r}\times\frac{d\vec{r}}{dt} + c$.

$\boxed{\text{Ex-1}}$: Evaluate $\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt$ where $\vec{r} = 2t^2\hat{i} + t\hat{j} - 3t^2\hat{k}$.

<u>Ans</u> : Since $\int \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \vec{r} \times \frac{d\vec{r}}{dt} + c$.

$$\therefore \int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \left[(2t^2\hat{i} + t\hat{j} - 3t^2\hat{k}) \times (4t\hat{i} + \hat{j} - 6t\hat{k})\right]_1^2$$

$$= \left[-3t^2\hat{i} - 2t^2\hat{k}\right]_1^2$$

$$= -9\hat{i} - 6\hat{k} = -3(3\hat{i} + 2\hat{k}).$$

$\boxed{\text{Ex-2}}$: If $\vec{r} \times \ddot{\vec{r}} = \vec{0}$ then show that $\vec{r} \times \dot{\vec{r}} = c$, where $\vec{r}$ is a vector function of a scalar variable t and $\vec{c}$ is a constant vector.

<u>Ans</u> : $\frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) = \vec{r} \times \ddot{\vec{r}} + \dot{\vec{r}} \times \dot{\vec{r}}$

$= \vec{r} \times \ddot{\vec{r}}$ $\quad [\because \dot{\vec{r}} \times \dot{\vec{r}} = \vec{0}]$

$= \vec{0}$ $(\vec{r} \times \ddot{\vec{r}} = \vec{0})$.

Integrating, $\vec{r} \times \dot{\vec{r}} = \vec{c}$, $\vec{c}$ is constant vector.

$\boxed{\text{Ex-3}}$: Find the value of $\vec{r}$ satisfying the equation $\frac{d^2\vec{r}}{dt^2} = \vec{a}$, where $\vec{a}$ is a constant vector. Hence show that $\vec{r} = \frac{1}{2}\vec{a}t^2 + ut$ if $\vec{r} = \vec{0}$ when $t = 0$ and $\frac{d\vec{r}}{dt} = \vec{u}$ when $t = 0$.

<u>Ans</u> : Given equation is $\frac{d^2\vec{r}}{dt^2} = \vec{a}$.

Integrating, $\frac{d\vec{r}}{dt} = \vec{a}t + \vec{c}$.

Initially, $t=0$, $\frac{d\vec{r}}{dt} = \vec{u}$, then $\vec{c} = \vec{u}$.

$$\therefore \frac{d\vec{r}}{dt} = \vec{a}t + \vec{u}.$$

Again integrating $\vec{r} = \vec{a}\frac{t^2}{2} + \vec{u}t + \vec{c_1}$.

Initially, $t=0$, $\vec{r} = \vec{0}$. $\vec{c_1} = \vec{0}$.

$\therefore \quad \vec{r} = \frac{1}{2}\vec{a}t^2 + \vec{u}t$.

Ex-3(b). Find $\vec{r}$ from the equation $\frac{d^2\vec{r}}{dt^2} = \vec{a}t + \vec{b}$ where $\vec{a}$ and $\vec{b}$ are constant vectors, given that $\vec{r}$ and $\frac{d\vec{r}}{dt}$ vanish when $t=0$.

Ans. Given equation is $\frac{d^2\vec{r}}{dt^2} = \vec{a}t + \vec{b}$.

Integrating, $\frac{d\vec{r}}{dt} = \vec{a}\frac{t^2}{2} + \vec{b}t + \vec{c}$.

Initially, $t=0$, $\frac{d\vec{r}}{dt} = 0 \quad \therefore \vec{c} = \vec{0}$.

$$\therefore \frac{d\vec{r}}{dt} = \vec{a}\frac{t^2}{2} + \vec{b}t$$

Again integrating, $\vec{r} = \vec{a}\frac{t^3}{6} + \vec{b}\frac{t^2}{2} + \vec{c_1}$.

~~Initially~~, when $t=0$, $\vec{r} = \vec{0}$, $\therefore \vec{c_1} = \vec{0}$

$$\therefore \vec{r} = \vec{a}\frac{t^3}{6} + \vec{b}\frac{t^2}{2}$$

$\boxed{\text{Ex-3(c)}}$: If $\frac{d^2\vec{r}}{dt^2} = 6t\hat{i} - 24t^2\hat{j} + 4\sin t\,\hat{k}$ and if $\vec{r} = 2\hat{i} + \hat{j}$ and $\frac{d\vec{r}}{dt} = -\hat{i} - 3\hat{k}$ when $t=0$. Then show that $\vec{r} = (t^3 - t + 2)\hat{i} + (1 - 2t^4)\hat{j} + (t - 4\sin t)\hat{k}$.

Ans: Given equation is $\frac{d^2\vec{r}}{dt^2} = 6t\hat{i} - 24t^2\hat{j} + 4\sin t\,\hat{k}$.

Integrating, $\frac{d\vec{r}}{dt} = 3t^2\hat{i} - \frac{24t^3}{3}\hat{j} - 4\cos t\,\hat{k} + \vec{c}$.

When $t=0$, $\frac{d\vec{r}}{dt} = -\hat{i} - 3\hat{k}$, then

$$-\hat{i} - 3\hat{k} + 4\hat{k} = \vec{c}$$

$$\therefore \frac{d\vec{r}}{dt} = 3t^2\hat{i} - 8t^3\hat{j} - 4\cos t\,\hat{k} - \hat{i} - 3\hat{k} + 4\hat{k}$$

$$= 3t^2\hat{i} - 8t^3\hat{j} - 4\cos t\,\hat{k} - \hat{i} + \hat{k}$$

Again integrating, $\vec{r} = t^3\hat{i} - t\hat{i} - 2t^4\hat{j} - 4\sin t\,\hat{k} + t\hat{k} + \vec{d}$.

, $\vec{d}$ being constant vector.

When $t=0$, $\vec{r} = 2\hat{i}+\hat{j}$, $\quad 2\hat{i}+\hat{j} = \vec{d}$.

$$\therefore \vec{r} = t^3\hat{i} - t\hat{i} - 2t^4\hat{j} - 4\sin t\,\hat{k} + t\hat{k} + 2\hat{i} + \hat{j}$$

$$= (t^3 - t + 2)\hat{i} + (1-2t^4)\hat{j} + (t - 4\sin t)\hat{k}.$$

(Proved)

$\boxed{\text{Ex-4}}$: (a) The position vector $\vec{r}$ of a moving particle at time t satisfies the equation $\frac{d^2\vec{r}}{dt^2} + 4\vec{r} = \vec{0}$. Find $\vec{r}$ at any time subject to $\vec{r} = \vec{a}$ and $\frac{d\vec{r}}{dt} = \vec{0}$ when $t=0$.

(b) Find $\vec{r}$ satisfying the equation $\frac{d^2\vec{r}}{dt^2} + \omega^2\vec{r} = \vec{0}$ where ω is a constant different from zero, given that $\vec{r} = (1,1,0)$ and $\frac{d\vec{r}}{dt} = (0,0,1)$ when $t=0$.

Ans: Given equation is $\frac{d^2\vec{r}}{dt^2} = -4\vec{r}$.

Multiplying both sides by $2\frac{d\vec{r}}{dt}$ we have

$$2\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2} = -4\vec{r}\cdot 2\frac{d\vec{r}}{dt}.$$

$$\Rightarrow \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)^2 = -8\,\vec{r}\cdot\frac{d\vec{r}}{dt}.$$

Integrating, $\left(\frac{d\vec{r}}{dt}\right)^2 = -4\vec{r}^{\,2} + \vec{c}$.

When $\vec{r} = \vec{a}$, $\frac{d\vec{r}}{dt} = 0$ at $t=0$.

$\therefore \vec{c} = 4\vec{a}^{\,2}$

$$\therefore \left(\frac{d\vec{r}}{dt}\right)^2 = -4\vec{r}^{\,2} + 4\vec{a}^{\,2}$$

$$\therefore \frac{d\vec{r}}{dt} = 2\sqrt{\vec{a}^{\,2} - \vec{r}^{\,2}}.$$

$$\Rightarrow \frac{d\vec{r}}{\sqrt{\vec{a}^{\,2} - \vec{r}^{\,2}}} = 2dt.$$

Integrating $\sin^{-1}\left(\frac{\vec{r}}{\vec{a}}\right) = 2t + \vec{B}$; $\vec{D}$ being constant.

When $t=0$, $\vec{r} = \vec{a}$, $\vec{B} = \pi/2$

$$\therefore \frac{\vec{r}}{\vec{a}} = \sin(2t + \pi/2)$$

$$\boxed{\Rightarrow \vec{r} = \vec{a}\cos 2t}.$$

(6) Try your self :

Ex-5: The acceleration of a particle at any time $t \geq 0$ is given by $\vec{a} = \frac{d\vec{v}}{dt} = 12\cos 2t\,\hat{i} - 8\sin 2t\,\hat{j} + 16t\,\hat{k}$.
If the velocity $\vec{v}$ and displacement $\vec{r}$ are zero at $t=0$, find $\vec{v}$ and $\vec{r}$ at any time.

Ans: Given that $\frac{d\vec{v}}{dt} = 12\cos 2t\,\hat{i} - 8\sin 2t\,\hat{j} + 16t\,\hat{k}$

Integrating, $\vec{v} = 6\sin 2t\,\hat{i} + 4\cos 2t\,\hat{j} + 8t^2\hat{k} + \vec{c_1}$.

Putting $\vec{v} = \vec{0}$ when $t=0$ $\therefore \vec{c_1} = -4\hat{j}$.

$$\frac{d\vec{r}}{dt} = \vec{v} = 6\sin 2t\,\hat{i} + (4\cos 2t - 4)\hat{j} + 8t^2\hat{k}.$$

$\therefore$ Integrating, $\vec{r} = -3\cos 2t\,\hat{i} + (2\sin 2t - 4t)\hat{j} + \frac{8}{3}t^3\hat{k} + \vec{c_2}$

Putting $\vec{r} = 0$ when $t=0$ $\therefore \vec{c_2} = 3\hat{i}$.

$$\therefore \vec{r} = (3 - 3\cos 2t)\hat{i} + (2\sin 2t - 4t)\hat{j} + \frac{8}{3}t^3\hat{k}.$$

Ex-6: The equation of motion of a particle of mass m is given by $m\frac{d^2\vec{r}}{dt^2} = f(r)\,\vec{r_1}$ where $\vec{r}$ is the position vector of P measured from an origin O. $\vec{r_1}$ is a unit vector in the direction $\vec{r}$ and $f(r)$ is a function of the distance of P from O.

(a) Show that $\vec{r} \times \frac{d\vec{r}}{dt} = \vec{c}$ where $\vec{c}$ being constant vector.

(b) Interpret physically the cases $f(r) < 0$ and $f(r) > 0$.

(c) Interpret the result in (a) geometrically.

Ans: (a) Given equation $m \frac{d^2\vec{r}}{dt^2} = f(r)\, \vec{r_1}$

$$m\, \vec{r} \times \frac{d^2\vec{r}}{dt^2} = f(r)\, \vec{r} \times \vec{r_1}$$ [Multiplying vectorically by $\vec{r}$]

$$\Rightarrow m\, \vec{r} \times \frac{d^2\vec{r}}{dt^2} = \vec{0}$$ [$\vec{0}$ is null vector]

[$\because$ Since $\vec{r}$ and $\vec{r_1}$ are collinear so, $\vec{r} \times \vec{r_1} = \vec{0}$]

$$\Rightarrow \frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \vec{0}.$$

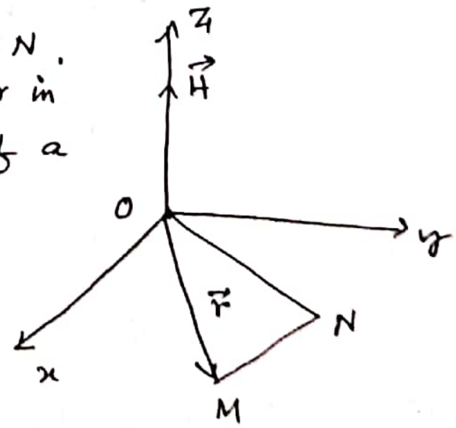
Integrating, $\vec{r} \times \frac{d\vec{r}}{dt} = \vec{c}$, $\vec{c}$ being constant vector.

(b) If $f(r) < 0$, the acceleration $\frac{d^2\vec{r}}{dt^2}$ has direction opposite to $\vec{r_1}$; hence the force is directed toward O and the particle is always attracted toward O.

If $f(r) > 0$ the force is directed away from O and the particle is under the influence of a repulsive force at O.

A force directed toward or away from a fixed point O and having magnitude depending only the distance r from O is called a central force.

(c) In time Δt, the particle moves from M to N. The area swept out by the position vector in this time is approximately half the area of a parallelogram with sides $\vec{r}$ and $\Delta\vec{r}$ or $\frac{1}{2}\vec{r} \times \Delta\vec{r}$.



The approximate area swept out by the radius vector per unit time is $\frac{1}{2}\vec{r} \times \frac{\Delta\vec{r}}{\Delta t}$.

The instantaneous rate of change in area is $\lim_{\Delta t \to 0} \frac{1}{2}\vec{r} \times \frac{\Delta\vec{r}}{\Delta t} = \frac{1}{2}\vec{r} \times \vec{v}$.

The quantity $\vec{H} = \frac{1}{2}\vec{r} \times \vec{v}$ is called the areal velocity.

Areal velocity $= \vec{H} = \frac{1}{2}\vec{r} \times \frac{d\vec{r}}{dt} =$ constant.