Vector Differentiation:

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Paper- C201

Course- Mathematics (H)

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Vector Analysis:

Vector Function: If to each value of a scalar variable t in some interval (a,b) or $[a,b]$, there corresponds a unique vector $\vec{f}(t)$, then $\vec{f}(t)$ is called a vector function of t.

Let $\vec{r} = \vec{OP}$ be the vector from a fixed origin O to a variable point P on the curve in space. If there exits an independent scalar variable t such that corresponding to each value of t, we get a definite position of P i.e. a unique vector $\vec{r}$, $\vec{r}$ is called a single valued vector function of the scalar variable t in this interval ∴ We write as $\vec{r} = \vec{f}(t)$.

If $\hat{i}, \hat{j}, \hat{k}$ be three unit vectors along three mutually fixed direction, it is possible to decompose $\vec{f}(t)$ as

$$\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}.$$

Where $f_1(t)$, $f_2(t)$, $f_3(t)$ are scalar function of t.

$$\therefore \vec{f}(t) = (f_1(t), f_2, f_3).$$

For ex: (i) $\vec{r} = \vec{f}(\alpha) = a\cos\alpha\,\hat{i} + b\sin\alpha\,\hat{j} + 0\hat{k}$, α being the scalar variable, is the vector equation of an ellipse with $2a$ and $2b$ as the major and minor axes resp.

(ii) $\vec{r} = \vec{f}(t) = \cos t\,\hat{i} + \sin t\,\hat{t} + 0.\hat{k}$ is a vector equation of circle of unit radius.

Limit of vector function:

A vector function $\vec{f}(t)$ is said to tend to a limit $\vec{A}$ as $t \to t_0$ if to any pre assigned positive number ϵ, however small, there corresponds a positive number δ such that

$$|\vec{f}(t) - \vec{A}| < \epsilon \text{ when } 0 < |t - t_0| < \delta.$$

Also, we may write $\lim_{t \to t_0} \vec{F}(t) = \vec{\lambda}$.

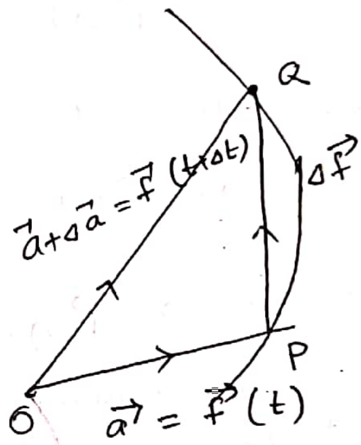
Continuity of a vector function:

Let $\vec{f}(t)$ denote the vector function of a scalar variable t over the interval $a \le t \le b$. The vector function $\vec{f}(t)$ is said to be continuous at $t = t_0$ if
$$\lim_{t \to t_0} \vec{f}(t) = \vec{f}(t_0).$$

Otherwise,

The vector function $\vec{f}(t)$ is said to be continuous at $t = t_0$ if given any positive number ϵ, it is possible to find another positive number δ such that $|\vec{f}(t) - \vec{f}(t_0)| < \epsilon$ whenever $|t - t_0| < \delta$.

Derivative of a vector function:



Suppose $\vec{f}(t)$ be the single valued function of a scalar variable t. Let the position vector of a point P relative to fixed point O be $\vec{a}$ on the curve be continuous and single valued function of a scalar variable t

i.e. $\vec{OP} = \vec{a} = \vec{f}(t)$.

$\vec{a} + \Delta\vec{a}$ be the position vector of Q on the curve when the scalar variable t changes to $t + \Delta t$.

$$\therefore \vec{OQ} = \vec{a} + \Delta\vec{a} = \vec{f}(t + \Delta t).$$

$\therefore$ From figure, $\Delta\vec{a} = \vec{PQ} = \vec{f}(t + \Delta t) - \vec{f}(t)$.

$$\therefore \lim_{\Delta t \to 0} \frac{\Delta\vec{f}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{f}(t+\Delta t) - \vec{f}(t)}{\Delta t} \quad [\vec{PQ} = \Delta\vec{f}]$$

If limit exists, the function $\vec{f}(t)$ said to be the derivable at any point t in the interval of definition of the function,

$$\frac{d\vec{f}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{f}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{f}(t+\Delta t) - \vec{f}(t)}{\Delta t}.$$

Also, $\vec{f}'(t_0) = \lim_{\Delta t \to 0} \dfrac{\vec{f}(t_0+\Delta t) - \vec{f}(t_0)}{\Delta t}$ exists.

Theorem: If $\vec{f}'(t)$ exists at $t = t_0$ then $\vec{f}(t)$ is continuous at $t = t_0$. In other words, every derivable function is continuous. Is the converse true. Justify.

Proof: Since $\vec{f}'(t)$ exists at $t = t_0$,

$$\therefore \vec{f}'(t_0) = \lim_{\Delta t \to 0} \frac{\vec{f}(t_0+\Delta t) - \vec{f}(t_0)}{\Delta t} \quad -(1).$$

Now, $\lim_{\Delta t \to 0} \left[\vec{f}(t_0+\Delta t) - \vec{f}(t_0)\right]$

$$= \lim_{\Delta t \to 0} \left[\Delta t \left\{ \frac{\vec{f}(t_0+\Delta t) - \vec{f}(t_0)}{\Delta t} \right\}\right]$$

$$= \lim_{\Delta t \to 0} \Delta t \cdot \lim_{\Delta t \to 0} \frac{\vec{f}(t_0+\Delta t) - \vec{f}(t_0)}{\Delta t}.$$

$$= 0 \times \vec{f}'(t_0) = 0.$$

$$\therefore \lim_{\Delta t \to 0} \vec{f}(t_0+\Delta t) = \vec{f}(t_0).$$

$\Rightarrow \vec{f}(t)$ is continuous at $t = t_0$.

Converse: The converse of this theorem is not always true.

Let $\vec{f}(t) = |t|\,\hat{i}$.

Here $\vec{f}(t)$ is continuous at $t=0$, but not derivable at $t=0$.

Since $\left|\vec{f}(t)-\vec{f}(0)\right| = \left| |t|\hat{i} - 0\right| = |t|$.

$\therefore \lim_{t\to 0} \vec{f}(t) = 0 = \vec{f}(0)$.

So $\vec{f}(t)$ is continuous at $t=0$.

But $\lim_{t\to 0} \dfrac{\vec{f}(t)-\vec{f}(0)}{t-0}$

$= \lim_{t\to 0} \dfrac{|t|\hat{i}}{t}$

$= -\hat{i}$ as $t\to 0^-$

$= \hat{i}$ as $t\to 0^+$.

Hence $\vec{f}'(t)$ does not exist at $t=0$.

Since, the limit value is not unique,

Hence the theorem.

Differentiation Formula: If $\vec{a}, \vec{b}, \vec{c}$ are differentiable vector function of a scalar t and φ is a differentiable scalar function of t.

Then (i) $\dfrac{d}{dt}(\vec{a}+\vec{b}) = \dfrac{d\vec{a}}{dt} + \dfrac{d\vec{b}}{dt}$.

(ii) $\dfrac{d}{dt}(\varphi\vec{a}) = \varphi\dfrac{d\vec{a}}{dt} + \dfrac{d\varphi}{dt}\vec{a}$

(iii) $\dfrac{d}{dt}(\vec{a}\cdot\vec{b}) = \vec{a}\cdot\dfrac{d\vec{b}}{dt} + \dfrac{d\vec{a}}{dt}\cdot\vec{b}$

(iv) $\dfrac{d}{dt}(\vec{a}\times\vec{b}) = \vec{a}\times\dfrac{d\vec{b}}{dt} + \dfrac{d\vec{a}}{dt}\times\vec{b}$

(v) $\dfrac{d}{dt}(\vec{a}\cdot\vec{b}\times\vec{c}) = \vec{a}\cdot\vec{b}\times\dfrac{d\vec{c}}{dt} + \vec{a}\cdot\dfrac{d\vec{b}}{dt}\times\vec{c} + \dfrac{d\vec{a}}{dt}\cdot\vec{b}\times\vec{c}$

(vi) $\dfrac{d}{dt}(\vec{a}\times(\vec{b}\times\vec{c})) = \vec{a}\times\left(\vec{b}\times\dfrac{d\vec{c}}{dt}\right) + \vec{a}\times\left(\dfrac{d\vec{b}}{dt}\times\vec{c}\right) + \dfrac{d\vec{a}}{dt}\times(\vec{b}\times\vec{c})$.

Proof. (i) If $\Delta\vec{a}$ and $\Delta\vec{b}$ be the respective increments of two vectors $\vec{a}$ and $\vec{b}$ due to the increment Δt of t.

$$\therefore \Delta(\vec{a}+\vec{b}) = (\vec{a}+\Delta\vec{a}+\vec{b}+\Delta\vec{b}) - (\vec{a}+\vec{b})$$
$$= \Delta\vec{a} + \Delta\vec{b}$$

$$\therefore \lim_{\Delta t\to 0} \frac{\Delta(\vec{a}+\vec{b})}{\Delta t} = \lim_{\Delta t\to 0} \frac{\Delta\vec{a}}{\Delta t} + \lim_{\Delta t\to 0} \frac{\Delta\vec{b}}{\Delta t}$$

$$\Rightarrow \frac{d}{dt}(\vec{a}+\vec{b}) = \frac{d\vec{a}}{dt} + \frac{d\vec{b}}{dt}$$

(ii) $$\Delta(\phi\vec{a}) = (\phi+\Delta\phi)(\vec{a}+\Delta\vec{a}) - \phi\vec{a}$$
$$= \phi\,\Delta\vec{a} + \vec{a}\,\Delta\phi + \Delta\phi\,\Delta\vec{a}.$$

$$\therefore \frac{\Delta(\phi\vec{a})}{\Delta t} = \phi\frac{\Delta\vec{a}}{\Delta t} + \vec{a}\frac{\Delta\phi}{\Delta t} + \frac{\Delta\phi}{\Delta t}\Delta\vec{a}.$$

$$\lim_{\Delta t\to 0}\frac{\Delta(\phi\vec{a})}{\Delta t} = \lim_{\Delta t\to 0}\phi\frac{\Delta\vec{a}}{\Delta t} + \vec{a}\lim_{\Delta t\to 0}\frac{\Delta\phi}{dt} + 0 \quad [\Delta\phi, \Delta\vec{a}\to 0 \text{ as } \Delta t\to 0]$$

$$\Rightarrow \frac{d(\phi\vec{a})}{dt} = \phi\frac{d\vec{a}}{dt} + \vec{a}\frac{d\phi}{dt}.$$

(iii) Let $\Delta\vec{a}, \Delta\vec{b}$ be the increments of the vector $\vec{a}$ and $\vec{b}$ respectively due to the increment Δt of t.

$$\therefore \Delta(\vec{a}\cdot\vec{b}) = (\vec{a}+\Delta\vec{a})\cdot(\vec{b}+\Delta\vec{b}) - \vec{a}\cdot\vec{b}$$
$$= \vec{a}\cdot\Delta\vec{b} + \Delta\vec{a}\cdot\vec{b} + \Delta\vec{a}\cdot\Delta\vec{b} \quad [\text{Since dot product is distributive}]$$

Now $$\lim_{\Delta t\to 0}\frac{\Delta(\vec{a}\cdot\vec{b})}{\Delta t} = \lim_{\Delta t\to 0}\vec{a}\cdot\frac{\Delta\vec{b}}{\Delta t} + \lim_{\Delta t\to 0}\frac{\Delta\vec{a}}{\Delta t}\cdot\vec{b}$$
$$[\Delta\vec{a}, \Delta\vec{b}\to 0 \text{ as } \Delta t\to 0]$$

$$\therefore \frac{d}{dt}(\vec{a}\cdot\vec{b}) = \vec{a}\cdot\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\cdot\vec{b}.$$

<u>Another method</u>: Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.
$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

$$\therefore \frac{d}{dt}(\vec{a}\cdot\vec{b}) = \frac{d}{dt}(a_1b_1 + a_2b_2 + a_3b_3)$$

$$= \left(a_1\frac{db_1}{dt} + a_2\frac{db_2}{dt} + a_3\frac{db_3}{dt}\right)$$

$$+ \frac{da_1}{dt}b_1 + \frac{da_2}{dt}b_2 + \frac{da_3}{dt}b_3$$

$$= \vec{a}\cdot\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\cdot\vec{b}$$

(iv). Let $\Delta\vec{a}$, $\Delta\vec{b}$ be the respective increments of $\vec{a}$ and $\vec{b}$ corresponding to the increment Δt of t.

$$\therefore \Delta(\vec{a}\times\vec{b}) = (\vec{a}+\Delta\vec{a})\times(\vec{b}+\Delta\vec{b}) - \vec{a}\times\vec{b}$$

$$= \vec{a}\times\Delta\vec{b} + \Delta\vec{a}\times\vec{b} + \Delta\vec{a}\times\Delta\vec{b}.$$

[Vector product is distributive]

$$\Rightarrow \lim_{\Delta t\to 0}\frac{\Delta(\vec{a}\times\vec{b})}{\Delta t} = \lim_{\Delta t\to 0}\frac{\vec{a}\times\Delta\vec{b}}{\Delta t} + \lim_{\Delta t\to 0}\frac{\Delta\vec{a}}{\Delta t}\times\vec{b}$$

[$\Delta\vec{a}, \Delta\vec{b}\to 0$ as $\Delta t\to 0$]

$$\Rightarrow \frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{a}\times\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\times\vec{b}$$

__Another way:__ Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

$$\therefore \frac{d}{dt}(\vec{a}\times\vec{b}) = \frac{d}{dt}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ \frac{db_1}{dt} & \frac{db_2}{dt} & \frac{db_3}{dt}\end{vmatrix} + \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{da_1}{dt} & \frac{da_2}{dt} & \frac{da_3}{dt}\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$= \vec{a}\times\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\times\vec{b}.$$

Theorem : The necessary and sufficient condition for a vector function $\vec{f}(t)$ to be a constant is $\frac{d}{dt}(\vec{f}(t)) = \vec{0}$

Proof : <u>The condition is necessary :</u> If $\vec{f}(t)$ be a constant, then

We have to prove that $\frac{d\vec{f}(t)}{dt} = \vec{0}$

Since $\vec{f}(t)$ be constant, then for every change h of the scalar variable t, $\vec{f}(t+h) - \vec{f}(t) = \vec{0}$

$$\therefore \frac{d\vec{f}}{dt} = \lim_{h \to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h} = \vec{0}.$$

$$\Rightarrow \frac{d\vec{f}(t)}{dt} = \vec{0}.$$

<u>The condition is sufficient</u> : We assume that $\frac{d\vec{f}(t)}{dt} = 0$

To prove that $\vec{f}(t)$ is a constant.

Let $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$.

Where $f_1(t)$, $f_2(t)$, $f_3(t)$ are three scalar function of t.

$$\therefore \frac{d\vec{f}}{dt} = 0$$

$$\Rightarrow \frac{d\vec{f}_1(t)}{dt}\hat{i} + \frac{d\vec{f}_2(t)}{dt}\hat{j} + \frac{df_3(t)}{dt}\hat{k} = 0.$$

$$\Rightarrow \frac{df_1}{dt} = \frac{df_2}{dt} = \frac{df_3}{dt} = 0$$

and hence the scalar functions $f_1(t)$, $f_2(t)$, $f_3(t)$ are constant.

Hence $\vec{f}(t)$ is constant.

Theorem : The necessary and sufficient condition for a vector $\vec{r} = \vec{f}(t)$ to have a constant magnitude is $\vec{f} \cdot \frac{d\vec{f}}{dt} = 0$

Proof : <u>The condition is necessary :</u>

Let us assume that $\vec{f}(t)$ has a constant magnitude

$\because |\vec{f}| = \text{Constant}$

$\Rightarrow |\vec{f}|^2 = \text{Constant}$

$\Rightarrow \vec{f}\cdot\vec{f} = \text{Constant}$

$\Rightarrow \frac{d}{dt}(\vec{f}\cdot\vec{f}) = 0$

$\Rightarrow \vec{f}\cdot\frac{d\vec{f}}{dt} + \frac{d\vec{f}}{dt}\cdot\vec{f} = 0$

$\Rightarrow 2\vec{f}\cdot\frac{d\vec{f}}{dt} = 0$

$\Rightarrow \vec{f}\cdot\frac{d\vec{f}}{dt} = 0$

The condition is sufficient:

Let $\vec{f}$ be a vector such that

$$\vec{f}\cdot\frac{d\vec{f}}{dt} = 0$$

$$\Rightarrow 2\vec{f}\cdot\frac{d\vec{f}}{dt} = 0$$

$$\Rightarrow \vec{f}\cdot\frac{d\vec{f}}{dt} + \frac{d\vec{f}}{dt}\cdot\vec{f} = 0$$

$$\Rightarrow \frac{d}{dt}(\vec{f}\cdot\vec{f}) = 0$$

$$\Rightarrow \frac{d}{dt}(|\vec{f}|^2) = 0$$

$$\Rightarrow |\vec{f}|^2 = \text{Constant}$$

$$\Rightarrow |\vec{f}| = \text{Constant}.$$

Thus, $\vec{f}$ is a vector of constant magnitude.

Theorem: The necessary and sufficient condition for a vector $\vec{f} = \vec{f}(t)$ to have a constant direction is $\vec{f}\times\frac{d\vec{f}}{dt} = \vec{0}$

Proof: Let $g(t)$ be the magnitude of $\vec{f}(t)$ and $\vec{F}(t)$ be the vector function whose modulus is unity for all values of t. so that

$$\vec{f}(t) = g(t)\,\vec{F}(t).$$

Then $\frac{d\vec{f}}{dt} = g(t)\frac{d\vec{F}}{dt} + \frac{dg}{dt}\vec{F}.$

Multiplying vectorially by $\vec{f}(t)$ we have

$$\vec{f}\times\frac{d\vec{f}}{dt} = \vec{f}\times\left(g\frac{d\vec{F}}{dt}+\frac{dg}{dt}\vec{F}\right)$$

$$= g\vec{F}(t)\times\left(g\frac{d\vec{F}}{dt}+\frac{dg}{dt}\vec{F}\right) \quad [\vec{f}=g(t)\vec{F}]$$

$$= g^2\vec{F}\times\frac{d\vec{F}}{dt} \quad [\because \vec{F}\times\vec{F}=0]$$

$$-(1).$$

<u>Condition is necessary:</u>

Let the direction of $\vec{f}(t)$ be constant.

So, $\vec{F}$ is a constant vector (both in magnitude and direction)

we have $\frac{d\vec{F}}{dt}=0$

Hence from (1), $\vec{f}\times\frac{d\vec{f}}{dt}=\vec{0}$

Thus the condition is necessary

<u>The condition is sufficient:</u>

We assume that $\vec{f}\times\frac{d\vec{f}}{dt}=\vec{0}$ .....(2)

Then from (1), $g^2\vec{F}\times\frac{d\vec{F}}{dt}=\vec{0}$ ..... (3)

Since $g(t)$ is not always zero, we have from (3)

$$\vec{F}\times\frac{d\vec{F}}{dt}=\vec{0} \quad \cdots (4)$$

Also, $\vec{F}$ being a vector with unit modulus, then

$$\vec{F}\cdot\frac{d\vec{F}}{dt}=0 \quad \cdots\cdots (5)$$

From (4) and (5), we have $\frac{d\vec{F}}{dt}=0$.

$\Rightarrow$ $\vec{F}$ is a constant vector.

$\Rightarrow$ $\vec{f}(t)$ has a constant direction.

<u>Hence the theorem</u>

Application of Vectors to Differential Geometry :

Some Preliminaries :

Curve : A Curve is an aggregate of points whose Co-ordinates are functions of a single variable t, is called a parameter and is represented by $\vec{r} = \vec{f}(t)$, $\vec{r}$ is a position vector of a point on the curve.

Ordinary point : A parameter t corresponding to the point is Called a ordinary point of a curve when the derivatives of the Co-ordinates exist and are continuous.

Smooth Curve : A Curve is called smooth curve if it consists entirely of ordinary points.

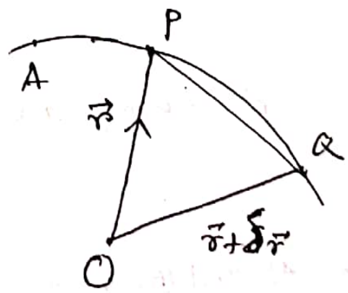
Plane Curve : A Curve is said to be plane curve if the set of points comprising a curve, lie on a plane.

Skew Curve : A curve is said to be skew curve if the set of points comprising a curve does not lie on a plane.

Unit Tangent Vector.

Let the smooth curve is

$\vec{r} = \vec{f}(s)$ where $AP = s$
$= \text{arclength}$



Let $P(s)$ and $Q(s+\delta s)$ be the points on the curve corresponding to the vectors $\vec{r}$ and $\vec{r}+\delta\vec{r}$, so $\vec{PQ} = \delta\vec{r}$

$$\therefore \frac{d\vec{r}}{ds} = \lim_{\delta s \to 0} \frac{\delta\vec{r}}{\delta s}$$

$$\therefore \boxed{\vec{t} = \frac{d\vec{r}}{ds}}$$

$\hat{t}(s)$ is Called the unit tangent vector.

with its direction along the tangent to the curve at P in increasing sense of s.

Let $\vec{r} = x(s)\,\hat{i} + y(s)\,\hat{j} + z(s)\,\hat{k}$.

$\therefore \vec{t} = \dfrac{d\vec{r}}{ds} = \dfrac{dx}{ds}\hat{i} + \dfrac{dy}{ds}\hat{j} + \dfrac{dz}{ds}\hat{k}$.

$\therefore$ The direction cosine of the direction of tangent are $\dfrac{dx}{ds}, \dfrac{dy}{ds}, \dfrac{dz}{ds}$.

The vector equation of the tangent line at P is $\boxed{\vec{R} = \vec{r} + u\vec{t}}$

i.e. $(\vec{R} - \vec{r}) \times \vec{t} = 0$.

$\therefore$ The normal plane which is perpendicular to the tangent at P

is $\boxed{(\vec{R} - \vec{r}) \cdot \vec{t} = 0}$

The unit tangent vector is given by $\vec{t} = \dfrac{d\vec{r}}{ds}$

$$= \frac{d\vec{r}}{dt}\frac{dt}{ds}$$

$$= \frac{\dfrac{d\vec{r}}{dt}}{\dfrac{ds}{dt}}$$

$$\boxed{\vec{t} = \frac{d\vec{r}}{dt} \Big/ \left|\frac{d\vec{r}}{dt}\right|}$$

$\left[\because \dfrac{d\vec{r}}{dt} = \dfrac{d\vec{r}}{ds} \cdot \dfrac{ds}{dt} = \vec{t}\dfrac{ds}{dt}\right.$

$\left.\therefore \left|\dfrac{d\vec{r}}{dt}\right| = \left|\dfrac{ds}{dt}\right|\right]$.

Principal Normal and Osculating plane

The unit tangent vector is of constant length.

So $\vec{t} \cdot \dfrac{d\vec{t}}{ds} = 0 \Rightarrow \dfrac{d\vec{t}}{ds}$ is perpendicular to the curve at P.

If $\vec{n}$ be the unit vector along the normal, then $\dfrac{d\vec{t}}{ds}$ and $\vec{n}$ are parallel.

$\therefore$ $\boxed{\frac{d\vec{t}}{ds} = K\vec{n}}$, K is the positive quantity, is called the Curvature and $K = \frac{1}{\rho}$ where ρ is the radius of Curvature.

Principal Normal: The straight line through P in the direction of $\vec{n}$ is called the principal normal.

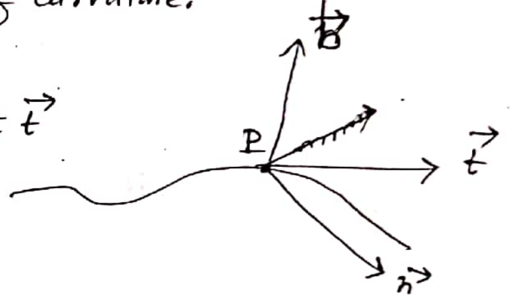
Its equation is $(\vec{R} - \vec{r}) \times \vec{n} = 0$

$$\text{i.e. } \boxed{\vec{R} = \vec{r} + u\vec{n}}$$

and $\vec{n}$ is called the unit principal normal.

Osculating plane (plane of Curvature):

The limiting position of the plane containing P and two adjacent points on the curve on either side of P as those points coincide with P, is called osculating plane or plane of curvature.



Osculating plane contains the unit tangent $\vec{t}$ and unit principal normal $\vec{n}$ at P.

The equation of the osculating plane is

$$[\vec{R} - \vec{r} \;\; \vec{t} \;\; \vec{n}] = 0$$

$$\Rightarrow (\vec{R} - \vec{r}) \cdot (\vec{t} \times \vec{n}) = 0$$

$$\Rightarrow \boxed{(\vec{R} - \vec{r}) \cdot \vec{b} = 0} \qquad \because [\vec{t} \times \vec{n} = \vec{b}]$$

Unit Binormal Vector:

The unit binormal vector is a vector $\vec{b}$ which is perpendicular to the osculating plane and is given by $\boxed{\vec{b} = \vec{t} \times \vec{n}}$.

Since $\vec{b}$ unit ~~unit~~ vector of constant magnitude.

$$\therefore \vec{b} \cdot \frac{d\vec{b}}{ds} = 0 \Rightarrow \frac{d\vec{b}}{ds} \text{ is } \perp^r \text{ to } \vec{b}.$$

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Also $\vec{b} = \vec{t} \times \vec{n}$

$$\frac{d\vec{b}}{ds} = \vec{t} \times \frac{d\vec{n}}{ds} + \frac{d\vec{t}}{ds} \times \vec{n}$$

$$= \vec{t} \times \frac{d\vec{n}}{ds} + \frac{\vec{n}}{\rho} \times \vec{n} \qquad \left[\because \frac{d\vec{t}}{ds} = \frac{1}{\rho}\vec{n}\right]$$

$$= \vec{t} \times \frac{d\vec{n}}{ds}.$$

$$\therefore \vec{t} \cdot \frac{d\vec{b}}{ds} = \vec{t} \cdot \left(\vec{t} \times \frac{d\vec{n}}{ds}\right)$$

$$= 0$$

$\therefore \frac{d\vec{b}}{ds}$ is $\perp^r$ to $\vec{t}$

Therefore, $\frac{d\vec{b}}{ds}$ is perpendicular to $\vec{b}$ and also to $\vec{t}$

$\therefore \frac{d\vec{b}}{ds}$ is parallel to $\vec{n}$.

$$\therefore \boxed{\frac{d\vec{b}}{ds} = -\tau \vec{n}}$$

Where τ is the torsion and $\frac{1}{\tau}$ is called the radius of torsion denoted by σ.

Torsion : Tortion is the are rate of turning of the unit binomial vector.

Rectifying plane: The plane perpendicular to the principal normal and containing the unit tangent and unit binormal vectors is called rectifying plane.

The equation of rectifying plane is

$$\boxed{(\vec{R} - \vec{r}) \cdot \vec{n} = 0}$$

Conclusion : $\vec{t}, \vec{n}, \vec{b}$ form a right handed system of mutually perpendicular unit vectors.

So, $\boxed{\vec{b} = \vec{t} \times \vec{n}, \quad \vec{n} = \vec{b} \times \vec{t} \quad \text{and} \quad \vec{t} = \vec{n} \times \vec{b}}$.

The triad formed by these vectors is called principal triad.

* ~~Ass~~ $\boxed{\text{Ex-1}}$: Prove the Frenet - Serret formulae

(i) $\frac{d\vec{t}}{ds} = k\vec{n}$

(ii) $\frac{d\vec{b}}{ds} = -\gamma\vec{n}$

(iii) $\frac{d\vec{n}}{ds} = \gamma\vec{b} - k\vec{t}$.

Where $\vec{t}, \vec{b}, \vec{n}$ are unit tangent, binormal and principal normal resp.

Proof: (i) Since $\vec{t}$ is a unit tangent vector.

So, $\vec{t}\cdot\vec{t} = |\vec{t}|^2 = 1$.

Differentiating with respect to the arc length s where

$$\frac{d}{ds}(\vec{t}\cdot\vec{t}) = 0$$

$$\Rightarrow \vec{t}\cdot\frac{d\vec{t}}{ds} + \vec{t}\cdot\frac{d\vec{t}}{ds} = 0$$

$$\Rightarrow \vec{t}\cdot\frac{d\vec{t}}{ds} = 0$$

$$\Rightarrow \frac{d\vec{t}}{ds} \text{ is perpendicular to } \vec{t}$$

If $\vec{n}$ be the unit normal vectors in the direction $\frac{d\vec{t}}{ds}$.

then $\frac{d\vec{t}}{ds}$ is parallel to $\vec{n}$

$\therefore \frac{d\vec{t}}{ds} = k\vec{n}$, $\vec{n}$ is the unit principal normal and k be the curvature of the curve.

(ii) Let $\vec{b} = \vec{t} \times \vec{n}$. where $\vec{b}$ is the unit binormal vectors. perpendicular to the plane containing $\vec{t}$ and $\vec{n}$.

Since $\vec{b}$ is a unit vector of constant magnitude.

$\therefore \vec{b}\cdot\vec{b} = 1$.

$\Rightarrow \vec{b}\cdot\frac{d\vec{b}}{ds} = 0$.

So, $\frac{d\vec{b}}{ds}$ is $\perp^r$ to $\vec{b}$.

Also, $\vec{b} = \vec{t} \times \vec{n}$.

Differentiating on both sides w.r.to s we have

$$\frac{d\vec{b}}{ds} = \vec{t} \times \frac{d\vec{n}}{ds} + \frac{d\vec{t}}{ds} \times \vec{n}$$

$$= \vec{t} \times \frac{d\vec{n}}{ds} + \frac{1}{\rho}\vec{n} \times \vec{n} \quad \left[\because \frac{d\vec{t}}{ds} = \frac{1}{\rho}\vec{n}\right]$$

$$= \vec{t} \times \frac{d\vec{n}}{ds} \quad [\because \vec{n} \times \vec{n} = 0]$$

$$\therefore \quad \vec{t} \cdot \frac{d\vec{b}}{ds} = 0.$$

$\therefore \Rightarrow \frac{d\vec{b}}{ds}$ is $\perp^r$ to $\vec{t}$.

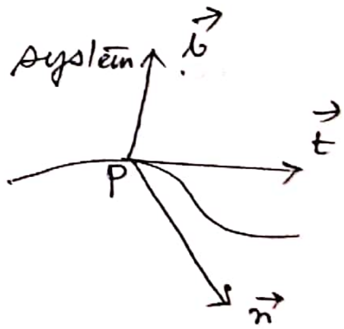
Therefore, $\frac{d\vec{b}}{ds}$ is $\perp^r$ to $\vec{b}$ and also to $\vec{t}$

$\therefore \frac{d\vec{b}}{ds}$ is parallel to $\vec{n}$.

$\therefore \frac{d\vec{b}}{ds} = -\tau\vec{n}$, τ is the torsion

$\sigma = \frac{1}{\tau}$ is the radius of torsion.

(iii) Since $\vec{t}, \vec{n}, \vec{b}$ form a right handed system:



So, $\vec{n} = \vec{b} \times \vec{t}$.

$$\frac{d\vec{n}}{ds} = \vec{b} \times \frac{d\vec{t}}{ds} + \frac{d\vec{b}}{ds} \times \vec{t}$$

$$= \vec{b} \times K\vec{n} + (-\tau\vec{n}) \times t$$

$$= K\,\vec{b} \times \vec{n} - \tau\,\vec{n} \times \vec{t}$$

$$= \tau\vec{b} - K\vec{t}. \quad \left[\because \vec{t} \times \vec{n} = \vec{b},\ \vec{n} \times \vec{b} = \vec{t}\right]$$

Hence the proof

Ex-2: Show that Frenet-Serret formulae can be written in the form $\frac{d\vec{t}}{ds} = \vec{\omega} \times \vec{t}$, $\frac{d\vec{n}}{ds} = \vec{\omega} \times \vec{n}$, $\frac{d\vec{b}}{ds} = \vec{\omega} \times \vec{b}$ and determine $\vec{\omega}$.

Ans : The Frenet - Serret formulae are

$$\frac{d\vec{t}}{ds} = k\vec{n}, \quad \frac{d\vec{b}}{ds} = -\tau\vec{n}, \quad \frac{d\vec{n}}{ds} = \tau\vec{b} - k\vec{t}.$$

Now

$$\begin{aligned}\frac{d\vec{n}}{ds} &= \tau\vec{b} - k\vec{t} \\ &= \tau(\vec{t}\times\vec{n}) - k(\vec{n}\times\vec{b}) \\ &= \tau(\vec{t}\times\vec{n}) + k(\vec{b}\times\vec{n}) = (\tau\vec{t} + k\vec{b})\times\vec{n} \\ &= \vec{\omega}\times\vec{n} \quad \text{where } \vec{\omega} = \tau\vec{t} + k\vec{b}.\end{aligned}$$

$$\begin{aligned}\frac{d\vec{t}}{ds} &= k\vec{n} = k(\vec{b}\times\vec{t}) + \tau(\vec{t}\times\vec{t}) \\ &= (k\vec{b} + \tau\vec{t})\times\vec{t} \\ &= \vec{\omega}\times\vec{t}.\end{aligned}$$

and

$$\begin{aligned}\frac{d\vec{b}}{ds} &= -\tau\vec{n} \\ &= -\tau\,\vec{b}\times\vec{t} + k\,\vec{b}\times\vec{b} \\ &= (\tau\vec{t} + k\vec{b})\times\vec{b} \\ &= \vec{\omega}\times\vec{b}.\end{aligned}$$

Therefore, the Frenet-Serret formulae can be written as

$$\frac{d\vec{t}}{ds} = \vec{\omega}\times\vec{t}, \quad \frac{d\vec{b}}{ds} = \vec{\omega}\times\vec{b}$$

and $\frac{d\vec{n}}{ds} = \vec{\omega}\times\vec{n}$ where $\vec{\omega} = \tau\vec{t} + k\vec{b}$.

$\boxed{\text{Ex-3}}$: If the equation of a space curve is given by $\vec{r} = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j} + b\theta\,\hat{k}$. Show that $k = \frac{a}{a^2+b^2}$ and $\tau = \frac{b}{a^2+b^2}$ where k and τ are curvature and torsion of the curve respectively.

Ans : Given curve is $\vec{r} = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j} + b\theta\,\hat{k}$.

$$\frac{d\vec{r}}{d\theta} = -a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j} + b\hat{k}.$$

$$\dot{s} = \frac{ds}{d\theta} = \left|\frac{d\vec{r}}{d\theta}\right| = \sqrt{a^2\sin^2\theta + a^2\cos^2\theta + b^2}$$

$= \sqrt{a^2+b^2}$

$\therefore \vec{t} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{d\theta} \cdot \frac{d\theta}{ds}$

$= \frac{1}{\sqrt{a^2+b^2}} \left(-a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j} + b\hat{k}\right)$.

$\frac{d\vec{t}}{d\theta} = \frac{1}{\sqrt{a^2+b^2}} \left(-a\cos\theta\,\hat{i} - a\sin\theta\,\hat{j}\right)$.

$\therefore \frac{d\vec{t}}{ds} = \frac{d\vec{t}}{d\theta} \cdot \frac{d\theta}{ds} = \frac{-a}{a^2+b^2} \left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right)$.

We have $\frac{d\vec{t}}{ds} = K\vec{n}$

$\therefore \left|\frac{d\vec{t}}{ds}\right| = K$

$\therefore K = \frac{a}{a^2+b^2} \sqrt{\sin^2\theta + \cos^2\theta} = \frac{a}{a^2+b^2}$

$\therefore \vec{n} = -\left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right)$.

Now, $\vec{b} = \vec{t} \times \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{-a\sin\theta}{\sqrt{a^2+b^2}} & \frac{a\cos\theta}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ -\cos\theta & -\sin\theta & 0 \end{vmatrix}$

$= \frac{\hat{i}\, b\sin\theta}{\sqrt{a^2+b^2}} - \frac{b\cos\theta}{\sqrt{a^2+b^2}}\hat{j} + \frac{a}{\sqrt{a^2+b^2}}\hat{k}$.

$\therefore \frac{d\vec{b}}{ds} = \frac{d\vec{b}}{d\theta} \cdot \frac{d\theta}{ds} = \frac{b}{\sqrt{a^2+b^2}} \left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right) \cdot \frac{1}{\sqrt{a^2+b^2}}$

$= \frac{b}{a^2+b^2} \left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right)$.

$\therefore \frac{d\vec{b}}{ds} = -\tau\vec{n} \quad \therefore \tau = \left|\frac{d\vec{b}}{ds}\right| = \frac{b}{a^2+b^2}$.

$\therefore K = \frac{a}{b^2+a^2}$ and $\tau = \frac{b}{a^2+b^2}$ (Proved).

$\boxed{\text{Ex-4}}$: (a) Prove that the curvature of the space curve $\vec{r} = \vec{r}(t)$ is given by numerically by $K = \dfrac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{r}|^3}$ where dot denotes the differentiation w.r.to t.

(b) Prove that $\tau = \dfrac{\dot{\vec{r}} \cdot \ddot{\vec{r}} \times \dddot{\vec{r}}}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2} = \dfrac{[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$ for the space curve $\vec{r} = \vec{r}(t)$.

V.H – 90, 90, 95

Ans : (a) Here $\vec{r} = \vec{r}(t)$, t being a scalar parameter.

Then $\dot{\vec{r}} = \dfrac{d\vec{r}}{dt} = \dfrac{d\vec{r}}{ds} \times \dfrac{ds}{dt} = \vec{t}\,\dot{s}$ ——(1).

$\therefore\ |\dot{\vec{r}}| = |\dot{s}|\quad [\because\ |\vec{t}| = 1]$.

Again, $\ddot{\vec{r}} = \dfrac{d\vec{t}}{dt}\dot{s} + \vec{t}\,\ddot{s} = \dfrac{d\vec{t}}{ds}\cdot\left(\dfrac{ds}{dt}\right)^2 + \vec{t}\,\ddot{s}$

$= \dfrac{\vec{n}}{\rho}\dot{s}^2 + \vec{t}\,\ddot{s}\quad \left[\dfrac{d\vec{t}}{ds} = K\vec{n} = \dfrac{1}{\rho}\vec{n}\right]$

——(2).

Thus, $\dot{\vec{r}} \times \ddot{\vec{r}} = \vec{t}\,\dot{s} \times \left(\dfrac{\vec{n}}{\rho}\dot{s}^2 + \vec{t}\,\ddot{s}\right)$

$= \dfrac{(\vec{t} \times \vec{n})\,\dot{s}^3}{\rho}$

$= \dfrac{\dot{s}^3\,\vec{b}}{\rho}\quad [\vec{t} \times \vec{n} = \vec{b}]$ ——(3).

From (3) $|\dot{\vec{r}} \times \ddot{\vec{r}}| = \dfrac{|\dot{s}|^3}{\rho}$

$\Rightarrow\ \dfrac{1}{\rho} = \dfrac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{s}|^3}$ ——(3.1)

$\Rightarrow\ K = \dfrac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{r}|^3}\quad \left[\ |\dot{\vec{r}}| = |\dot{s}| \text{ and } K = \dfrac{1}{\rho} = \text{Curvature}\right]$.

(Proved).

We have from (2), $\ddot{\vec{r}} = K\vec{n}\,\dot{s}^2 + \vec{t}\,\ddot{s}$

Differentiating (2) w.r.to t we have

$$\dddot{\vec{r}} = \frac{d\vec{n}}{ds}\cdot\frac{ds}{dt}\cdot k\dot{s}^2 + \frac{dk}{dt}\,\vec{n}\,\dot{s}^2 + k\vec{n}\,2\dot{s}\ddot{s} + \frac{d\vec{t}}{ds}\cdot\frac{ds}{dt}\,\ddot{s} + \vec{t}\,\dddot{s}$$

$$= \frac{d\vec{n}}{ds}\,k\dot{s}^3 + \frac{dk}{dt}\,\vec{n}\,\dot{s}^2 + 2k\vec{n}\,\dot{s}\ddot{s} + k\vec{n}\,\dot{s}\ddot{s} + \vec{t}\,\dddot{s} \quad \left[\because \frac{d\vec{t}}{ds} = k\vec{n}\right]$$

$$= (\tau\vec{b} - k\vec{t})\,k\dot{s}^3 + \frac{dk}{dt}\,\vec{n}\dot{s}^2 + 3k\vec{n}\,\dot{s}\ddot{s} + \vec{t}\,\dddot{s} \quad \left[\frac{d\vec{n}}{ds} = \tau\vec{b} - k\vec{t}\right]$$

$$= (\dddot{s} - k^2\dot{s}^3)\,\vec{t} + \left(\frac{dk}{dt}\,\dot{s}^2 + 3k\,\dot{s}\ddot{s}\right)\vec{n} + \tau k\dot{s}^3\vec{b} \quad \text{---(4)}$$

Now, $(\dot{\vec{r}} \times \ddot{\vec{r}})\cdot\dddot{\vec{r}} = \tau\,\dot{s}^6 k^2 \quad \left[\because \vec{t}\cdot\vec{b} = \vec{n}\cdot\vec{b} = 0 \text{ and } \vec{b}\cdot\vec{b} = 1\right]$

$$\Rightarrow \tau = \frac{[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}]}{|\dot{s}|^6 k^2}$$

$$= \frac{[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2} \quad [\text{using } (3\cdot 1)]$$

where τ = torsion and k = curvature, ρ = radius of curvature.

(Proved).

Note: Deduction of osculating plane, Normal plane and Rectifying plane.

We have from (1) and (3) $(\dot{\vec{r}} \times \ddot{\vec{r}}) \times \dot{\vec{r}} = \frac{\dot{s}^4}{\rho}(\vec{b} \times \vec{t})$

$$= \frac{\dot{s}^4}{\rho}\,\vec{n}. \quad \text{---(5)}$$

using (1), (3) and (5) we have

osculating plane is $(\vec{R} - \vec{r})\cdot\vec{b} = 0 \Rightarrow \boxed{(\vec{R} - \vec{r})\cdot(\dot{\vec{r}} \times \ddot{\vec{r}}) = 0}$

Normal plane is $(\vec{R}-\vec{r})\cdot\vec{t}=0 \Rightarrow \boxed{(\vec{R}-\vec{r})\cdot\dot{\vec{r}}=0}$

and Rectifying plane is $(\vec{R}-\vec{r})\cdot\vec{n} \Rightarrow \boxed{(\vec{R}-\vec{r})\cdot(\dot{\vec{r}}\times\ddot{\vec{r}})\times\dot{\vec{r}}=0}$

$\boxed{\text{Ex-5}}$: Prove that the torsion of the space curve $\vec{r}=\vec{r}(s)$ is given numerically by

$$\tau = \frac{\left|\frac{d\vec{r}}{ds}\cdot\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3}\right|}{\left|\frac{d^2\vec{r}}{ds^2}\right|^2}$$

i.e. $$\tau = \frac{\left[\frac{d\vec{r}}{ds}\ \frac{d^2\vec{r}}{ds^2}\ \frac{d^3\vec{r}}{ds^3}\right]\vec{n}^2}{\left(\frac{d^2\vec{r}}{ds^2}\right)^2}$$ V.H '90

Ans: We have $\vec{r}=\vec{r}(s)$

$\therefore \frac{d\vec{r}}{ds}=\vec{t}$

$\Rightarrow \frac{d^2\vec{r}}{ds^2}=\frac{d\vec{t}}{ds}=\kappa\vec{n}$ —(1) $\left[\frac{d\vec{t}}{ds}=\kappa\vec{n}\text{ by Frenet Serret formula}\right]$

Also, $\frac{d^3\vec{r}}{ds^3}=\frac{d\kappa}{ds}\vec{n}+\kappa\frac{d\vec{n}}{ds}$

$=\frac{d\kappa}{ds}\vec{n}+\kappa(\tau\vec{b}-\kappa\vec{t})$ $\left[\text{By Frenet Serret formula } \frac{d\vec{n}}{ds}=\tau\vec{b}-\kappa\vec{t}\right]$

$=\kappa\tau\vec{b}+\frac{d\kappa}{ds}\vec{n}-\kappa^2\vec{t}$.

Now, $\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3}=\kappa\vec{n}\times\left(\kappa\tau\vec{b}+\frac{d\kappa}{ds}\vec{n}-\kappa^2\vec{t}\right)$

$=\kappa^2\tau\,\vec{n}\times\vec{b}+\frac{d\kappa}{ds}\kappa\,\vec{n}\times\vec{n}-\kappa^3\,\vec{n}\times\vec{t}$

$=\kappa^2\tau\vec{t}+0+\kappa^3\vec{b}$ $\left[\because \vec{n}\times\vec{b}=\vec{t},\ \vec{n}\times\vec{n}=0,\ \vec{n}\times\vec{t}=-\vec{b}\right]$

$\frac{d\vec{r}}{ds}\cdot\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3}=\vec{t}\cdot(\kappa^2\tau\vec{t}+\kappa^3\vec{b})$

$=\kappa^2\tau$ —(2) $[\because \vec{t}\cdot\vec{t}=1 \text{ and } \vec{t}\cdot\vec{b}=0]$

$$\Rightarrow \left[\frac{d\vec{r}}{ds}\ \frac{d^2\vec{r}}{ds^2}\ \frac{d^3\vec{r}}{ds^3}\right] = \left(\frac{d^2\vec{r}}{ds^2}\right)^2 \frac{1}{\vec{n}^2}\cdot\tau \quad [\text{using } (1)]$$ (11)

$$\Rightarrow \tau = \frac{\left[\frac{d\vec{r}}{ds}\ \frac{d^2\vec{r}}{ds^2}\ \frac{d^3\vec{r}}{ds^3}\right]\vec{n}^2}{\left(\frac{d^2\vec{r}}{ds^2}\right)^2}.$$

Also from (2)

$$\therefore \tau = \frac{\left|\frac{d\vec{r}}{ds}\cdot\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3}\right|}{\left|\frac{d^2\vec{r}}{ds^2}\right|^2} \quad \left[\because \tau > 0 \text{ and } \vec{n}^2 = 1\right]$$

(Proved)

$\boxed{\text{Ex-6}}$: Prove that the following relations

(i) $\frac{1}{\rho} = |\vec{r}''|$ (ii) $\vec{r}'\times\vec{r}'' = \frac{1}{\rho}\vec{b}$ (iii) $\vec{r}''' = \frac{1}{\rho}\left(\tau\vec{b} - \frac{1}{\rho}\vec{t}\right) + k'\vec{n}$

and (iv) $[\vec{r}'\ \vec{r}''\ \vec{r}'''] = \frac{\tau}{\rho^2}$

Ans: In Ex-5, we have $\vec{r}' = \vec{r}(s)$

$$\frac{d\vec{r}}{ds} = \vec{t} \quad \text{—(1)}$$

$$\frac{d^2\vec{r}}{ds^2} = \frac{d\vec{t}}{ds} = k\vec{n} \quad \left[\frac{d\vec{t}}{ds} = k\vec{n}\right] \quad \text{—(2)}$$

$$\Rightarrow |\vec{r}''| = \frac{1}{\rho}|\vec{n}|$$

$$\Rightarrow \frac{1}{\rho} = |\vec{r}''| \left[\because |\vec{n}| = 1\right] \quad \text{(Proved)}$$

(ii)

$$\vec{r}'\times\vec{r}'' = k\,\vec{t}\times\vec{n}$$

$$= k\vec{b}$$

$$= \frac{1}{\rho}\vec{b} \quad \left[\because \vec{t}\times\vec{n} = \vec{b}\right]$$

(Proved)

(iii) Diff. (2) w. r. to s we have

$$\frac{d^3\vec{r}}{ds^3} = \frac{dk}{ds}\vec{n} + k\frac{d\vec{n}}{ds}$$

$$= \frac{dk}{ds}\vec{n} + k(\tau\vec{b} - k\vec{t}) \quad \left[\frac{d\vec{n}}{ds} = \tau\vec{b} - k\vec{t}\right]$$

$$\Rightarrow \quad \vec{r}''' = k'\vec{n} + \frac{1}{\rho}\left(\tau\vec{b} - \frac{1}{\rho}\vec{t}\right) \quad \text{(Proved)}$$

(iii) Now, $\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3} = k^2\tau\vec{t} + k^3\vec{b}$ [See Ex-5]

$$\therefore \quad \frac{d\vec{r}}{ds} \cdot \frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3} = \vec{t} \cdot \left(k^2\tau\vec{t} + k^3\vec{b}\right)$$

$$= k^2\tau \quad [\because \vec{t}\cdot\vec{t} = 1 \text{ and } \vec{t}\cdot\vec{b} = 0]$$

$$\therefore \tau = \frac{[\vec{r}' \; \vec{r}'' \; \vec{r}''']}{k^2}$$

$$\Rightarrow [\vec{r}' \; \vec{r}'' \; \vec{r}'''] = \frac{\tau}{\rho^2} \quad \left[k = \frac{1}{\rho}\right] \quad \text{(Proved)}.$$

$\boxed{\text{Ex-7}}$: Show that the curvature is the same at all points of a curve whose vector equation is $\vec{r} = (4\cos t, 4\sin t, 2t)$.

V.H-94, 97.

Ans:

The position vector for any point on the curve is given by

$$\vec{r} = 4\cos t\,\hat{i} + 4\sin t\,\hat{j} + 2t\,\hat{k}.$$

$$\therefore \quad \vec{t} = \frac{d\vec{r}}{ds} = \left(-4\sin t\,\hat{i} + 4\cos t\,\hat{j} + 2\hat{k}\right)\frac{dt}{ds}$$

$$\therefore \quad |\vec{t}| = \left|-4\sin t\,\hat{i} + 4\cos t\,\hat{j} + 2\hat{k}\right|\left|\frac{dt}{ds}\right|.$$

$$\Rightarrow 1 = \left(4^2 + 2^2\right)\left(\frac{dt}{ds}\right)^2$$

$$\Rightarrow \quad \frac{dt}{ds} = \frac{1}{\sqrt{20}}.$$

$$\therefore \quad \vec{t} = \left(-4\sin t\,\hat{i} + 4\cos t\,\hat{j} + 2\hat{k}\right)\frac{1}{\sqrt{20}}$$

$$\therefore \quad \frac{d\vec{t}}{ds} = \left(-4\cos t\,\hat{i} - 4\sin t\,\hat{j}\right)\frac{dt}{ds}\cdot\frac{1}{\sqrt{20}}$$

$$= -\frac{4}{20}\left(\cos t\,\hat{i} + \sin t\,\hat{j}\right)$$

$$= -\frac{1}{5}\left(\cos t\,\hat{i} + \sin t\,\hat{j}\right).$$

We have Frenet Serret formula

$$\frac{d\vec{t}}{ds} = K\vec{n}$$

$$\Rightarrow \left|\frac{d\vec{t}}{ds}\right| = |K|\,|\vec{n}|$$

$$= K \quad [\text{as } K > 0 \text{ and } |\vec{n}| = 1]$$

$$\Rightarrow K = \left|-\frac{1}{5}(\cos t\,\hat{i} + \sin t\,\hat{j})\right|$$

$$= \frac{1}{5}$$

$\therefore$ The curvature of the curve is $\frac{1}{5}$ which is constant.

Hence, the curvature is same at every point of the curve

$$\vec{r} = (4\cos t,\ 4\sin t,\ 2t).$$

Hence the result

$\boxed{\text{Ex-7.1}}$: Show that the curvature is the same at all points of a curve

$$\vec{r} = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j} + b\theta\,\hat{k}.$$

Ans: See Ex-3: $K = \dfrac{a}{a^2+b^2}$ which is constant.

Therefore, the curvature is same at all points of the curve

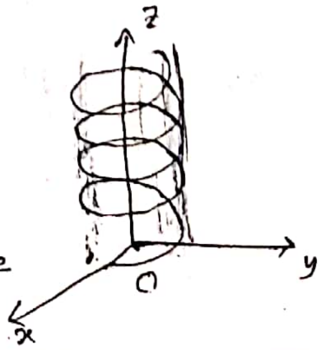
$$\vec{r} = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j} + b\theta\,\hat{k}.$$

$\boxed{\text{Ex-8}}$: Sketch the space curve $x = 3\cos t$, $y = 3\sin t$, $z = 4t$ and find (a) the unit tangent $\vec{t}$ (b) the principal normal $\vec{n}$, curvature K and radius of curvature ρ (c) the binormal $\vec{b}$, torsion τ and radius of torsion σ.

Ans: The space curve is a circular helix (see figure)

Since $t = z/4$, the curve has equations

$x = 3\cos(z/4)$, $y = 3\sin(z/4)$ and therefore lies on the cylinder $x^2+y^2=9$.



(a) The position vector for any point on the curve is $\vec{r} = 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + 4t\,\hat{k}$.

$$\frac{d\vec{r}}{dt} = -3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k}.$$

$$\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \left|-3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k}\right|$$

$$= \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2} = 5.$$

$$\therefore \vec{t} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds}$$

$$= \frac{1}{5}\left(-3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k}\right).$$

(c) $$\frac{d\vec{t}}{dt} = -\frac{3}{5}\cos t\,\hat{i} - \frac{3}{5}\sin t\,\hat{j}$$

$$\frac{d\vec{t}}{ds} = \frac{d\vec{t}}{dt} \times \frac{dt}{ds} = \frac{1}{5}\left(-\frac{3}{5}\cos t\,\hat{i} - \frac{3}{5}\sin t\,\hat{j}\right)$$

$$= -\frac{3}{25}\cos t\,\hat{i} - \frac{3}{25}\sin t\,\hat{j}.$$

We have by Frenet-Serret formula,

$$\frac{d\vec{t}}{ds} = \kappa\vec{n}$$

$$\Rightarrow \left|\frac{d\vec{t}}{ds}\right| = |\kappa|\,|\vec{n}|$$

$$= \kappa \text{ as } \kappa > 0 \text{ and } |\vec{n}| = 1.$$

$$\therefore \kappa = \left|-\frac{3}{25}\cos t\,\hat{i} - \frac{3}{25}\sin t\,\hat{j}\right|$$

$$= \sqrt{\left(-\frac{3}{25}\right)^2\cos^2 t + \left(\frac{3}{25}\sin t\right)^2} = \frac{3}{25}$$

$$\therefore \rho = \frac{1}{\kappa} = \frac{25}{3}.$$

Also, $$\frac{d\vec{t}}{ds} = \kappa\vec{n}$$

$$\Rightarrow \vec{n} = \frac{1}{\kappa}\frac{d\vec{t}}{ds} = \frac{25}{3}\left(-\frac{3}{25}\cos t\,\hat{i} - \frac{3}{25}\sin t\,\hat{j}\right)$$

$$= -\cos t\,\hat{i} - \sin t\,\hat{j}.$$

(c) Now, $\vec{b} = \vec{t} \times \vec{n}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{4}{5}\sin t\,\hat{i} - \frac{4}{5}\cos t\,\hat{j} + \frac{3}{5}\hat{k}.$$

$$\frac{d\vec{b}}{dt} = \frac{4}{5}\cos t\,\hat{i} + \frac{4}{5}\sin t\,\hat{j}.$$

$$\therefore \frac{d\vec{b}}{ds} = \frac{d\vec{b}}{dt}\frac{dt}{ds}$$

$$= \frac{1}{5}\left(\frac{4}{5}\cos t\,\hat{i} + \frac{4}{5}\sin t\,\hat{j}\right)$$

$$= \frac{4}{25}\left(\cos t\,\hat{i} + \sin t\,\hat{j}\right).$$

We have by Frenet-Serret formula

$$\frac{d\vec{b}}{ds} = -\tau\vec{n}$$

$$\Rightarrow \left|\frac{d\vec{b}}{ds}\right| = |\tau|\,|\vec{n}|$$

$$\Rightarrow \tau = \left|\frac{4}{25}\left(\cos t\,\hat{i} + \sin t\,\hat{j}\right)\right| = \frac{4}{25}$$

$$\therefore \sigma = \frac{1}{\tau} = \frac{25}{4}.$$

$\boxed{\text{Ex-9}}$: Prove that radius of curvature of the curve with parametric equations $x = x(s)$, $y = y(s)$, $z = z(s)$ is given by

$$\rho = \left[\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right]^{-1/2}.$$

Ans: The position vector of any point on the curve

$$\vec{r} = x(s)\,\hat{i} + y(s)\,\hat{j} + z(s)\,\hat{k}.$$

$$\therefore \vec{t} = \frac{d\vec{r}}{ds} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}.$$

$$\therefore \frac{d\vec{t}}{ds} = \frac{d^2x}{ds^2}\hat{i} + \frac{d^2y}{ds^2}\hat{j} + \frac{d^2z}{ds^2}\hat{k}.$$

By Frenet Serret formula,

$$\frac{d\vec{t}}{ds} = k\vec{n}, \quad k = \text{Curvature} = \frac{1}{\rho}.$$

$$\left|\frac{d\vec{t}}{ds}\right| = k \quad \left[\because |\vec{n}| = 1 \text{ and } k \geq 0\right]$$

$$\therefore \; k = \left| \frac{d^2x}{ds^2}\hat{i} + \frac{d^2y}{ds^2}\hat{j} + \frac{d^2z}{ds^2}\hat{k} \right|$$

$$= \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}.$$

$$\therefore \; \rho = \frac{1}{k} = \left[\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right]^{-1/2} \quad \text{(Proved)}.$$

$\boxed{\text{Ex-10}}$: Find $\vec{t}, \vec{b}, \vec{n}$ for the circular helix $\vec{r} = (a\cos\theta, a\sin\theta, a\theta\cot\beta)$. Find also expressions for curvature and torsion at a point on the curve.

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Ans: We have $\vec{t} = \frac{d\vec{r}}{ds}$

$$= \frac{d\vec{r}}{d\theta} \cdot \frac{d\theta}{ds}$$

$$= \left(-a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j} + a\cot\beta\,\hat{k}\right)\frac{d\theta}{ds}.$$

$$\Rightarrow \; |\vec{t}| = \left|-a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j} + a\cot\beta\,\hat{k}\right| \cdot \left|\frac{d\theta}{ds}\right|.$$

$$\Rightarrow \; \frac{d\theta}{ds} = \frac{1}{\sqrt{a^2 + a^2\cot^2\beta}} \quad \left[\because |\vec{t}| = 1\right]$$

$$= \frac{1}{a\operatorname{cosec}\beta} = \frac{\sin\beta}{a}$$

$$\therefore \; \vec{t} = \left(-a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j} + a\cot\beta\,\hat{k}\right)\frac{\sin\beta}{a}$$

$$= \sin\beta\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j} + \cot\beta\,\hat{k}\right). \quad \text{——(1)}.$$

Also, by Frenet Serret formula, $\frac{d\vec{t}}{ds} = k\vec{n}$ ——(2)

$$\Rightarrow \; k\vec{n} = \frac{d\vec{t}}{d\theta} \cdot \frac{d\theta}{ds} = \sin\beta\left(-\cos\theta\,\hat{i} - \sin\theta\,\hat{j}\right)\frac{\sin\beta}{a}.$$

$\therefore \quad |K\vec{n}| = \frac{\sin^2\beta}{a}\left(-\cos\theta\,\hat{i} - \sin\theta\,\hat{j}\right)$.

$\Rightarrow \quad K = \frac{\sin^2\beta}{a}$. = Curvature of the circular helix. ——(3)

$\therefore \quad \rho = \frac{1}{K} = \frac{a}{\sin\beta}$ which is the radius of curvature

Also from (2) $\quad \vec{n} = \frac{1}{K}\frac{d\vec{t}}{ds}$

$$= \frac{1}{K}\left(-\cos\theta\,\hat{i} - \sin\theta\,\hat{j}\right)\frac{\sin^2\beta}{a}$$

$$= -\left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right) \quad [\text{using } (3)]$$

$\therefore \quad \vec{n} = -\left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right)$. ——(4).

Also, $\quad \vec{b} = \vec{t}\times\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta\sin\beta & \cos\theta\sin\beta & \sin\beta\cot\beta \\ -\cos\theta & -\sin\theta & 0 \end{vmatrix}$

$$= \sin\beta\left[\sin\theta\cot\beta\,\hat{i} + \hat{j}\left(-\cos\theta\cot\beta\right) + \hat{k}\right]$$

$$= \sin\theta\cos\beta\,\hat{i} - \cos\beta\cos\theta\,\hat{j} + \hat{k}\sin\beta.$$

$\therefore \quad \frac{d\vec{b}}{ds} = \left(\cos\theta\cos\beta\,\hat{i} + \sin\theta\cos\beta\,\hat{j}\right)\cdot\frac{d\theta}{ds}$

$$= \frac{\sin\beta}{a}\left(\cos\theta\cos\beta\,\hat{i} + \sin\theta\cos\beta\,\hat{j}\right)$$

$$= \frac{\sin\beta\cos\beta}{a}\left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right).$$

We have by Frenet Serret formula,

$$\frac{d\vec{b}}{ds} = -\tau\vec{n}$$

$$\Rightarrow |\tau\vec{n}| = \left|\frac{d\vec{b}}{ds}\right|$$

$$\Rightarrow \tau = \frac{\sin\beta\cos\beta}{a}\sqrt{\sin^2\theta + \cos^2\theta} = \frac{\sin\beta\cos\beta}{a}$$

$\therefore$ Torsion is $\frac{\sin\beta\cos\beta}{a}$. $\quad [\because \tau > 0 \text{ and } |\vec{n}| = 1]$

Ex-11: If $\vec{r}$ is a unit vector then P.T $\left|\vec{r} \times \frac{d\vec{r}}{dt}\right| = \left|\frac{d\vec{r}}{dt}\right|$.

Ans: Since $\vec{r}$ is a unit vector

$$\therefore \vec{r}\cdot\vec{r} = 1$$

$$\frac{d}{dt}(\vec{r}\cdot\vec{r}) = 0$$

$$\Rightarrow 2\,\vec{r}\cdot\frac{d\vec{r}}{dt} = 0$$

$$\Rightarrow \vec{r}\cdot\frac{d\vec{r}}{dt} = 0.$$

$\therefore$ Two vectors $\vec{r}$ and $\frac{d\vec{r}}{dt}$ are at right angles provided

$$\left|\frac{d\vec{r}}{dt}\right| \neq 0$$

$$\therefore \vec{r} \times \frac{d\vec{r}}{dt} = |\vec{r}|\left|\frac{d\vec{r}}{dt}\right| \sin 90^\circ \hat{n}$$

$$= 1\left|\frac{d\vec{r}}{dt}\right| \hat{n}$$

$$\therefore \left|\vec{r} \times \frac{d\vec{r}}{dt}\right| = \left|\frac{d\vec{r}}{dt}\right| \quad [\because |\hat{n}| = 1].$$

Ex-12: If $\vec{R}$ be a unit vector in the direction of $\vec{r}$, Prove that

$$\vec{R} \times \frac{d\vec{R}}{dt} = \frac{1}{|\vec{r}|^2}\vec{r} \times \frac{d\vec{r}}{dt} \quad \text{ie} \quad \vec{R} \times \frac{d\vec{R}}{dt} = \frac{1}{r^2}\vec{r} \times \frac{d\vec{r}}{dt}$$

Where $|\vec{r}| = r$.

Ans: Given that $\vec{R} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}$

$$\therefore \frac{d\vec{R}}{dt} = \frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r}$$

$$\therefore \vec{R} \times \frac{d\vec{R}}{dt} = \frac{\vec{r}}{r} \times \left(\frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r}\right)$$

$$= \frac{1}{r^2}\vec{r} \times \frac{d\vec{r}}{dt} - \frac{1}{r^3}\frac{dr}{dt}\vec{r}\times\vec{r}$$

$$= \frac{1}{r^2}\vec{r} \times \frac{d\vec{r}}{dt} \quad [\because \vec{r}\times\vec{r} = 0]$$

Ex-13: Show that the curve $\vec{r} = \vec{r}(s)$ is a helix $\left(\frac{\kappa}{\tau} = \text{Constant}\right)$ iff $\frac{d\vec{r}}{ds^2} \cdot \frac{d^3\vec{r}}{ds^3} \times \frac{d^4\vec{r}}{ds^4} = 0$ Where s is the length of the Curve.

Ans : Given that $\vec{r} = \vec{r}(s)$

$$\frac{d\vec{r}}{ds} = \vec{t}$$

$$\frac{d^2\vec{r}}{ds^2} = \frac{d\vec{t}}{ds} = k\vec{n} \quad \text{---(1)} \quad \left[\text{By Frenet Serret formula, } \frac{d\vec{t}}{dt} = k\vec{n}\right]$$

$$\frac{d^3\vec{r}}{ds^3} = \frac{dk}{ds}\vec{n} + k\frac{d\vec{n}}{ds}$$

$$= \frac{dk}{ds}\vec{n} + k(\gamma\vec{b} - k\vec{t}) \quad \left[\because \text{By F-S formula, } \frac{d\vec{n}}{ds} = \gamma\vec{b} - k\vec{t}\right] \quad \text{---(2)}$$

$$= \frac{dk}{ds}\vec{n} + k\gamma\vec{b} - k^2\vec{t}.$$

Diff. (2) again w.r.to s we have

$$\frac{d^4\vec{r}}{ds^4} = \frac{d^2k}{ds^2}\vec{n} + \frac{dk}{ds}(\gamma\vec{b} - k\vec{t}) + \frac{dk}{ds}\gamma\vec{b}$$

$$+ k\frac{d\gamma}{ds}\vec{b} + k\gamma(-\gamma\vec{n}) - 2k\frac{dk}{ds}\vec{t}$$

$$- k^2 k\vec{n}$$

$$\left[\because \frac{d\vec{t}}{ds} = k\vec{n}, \ \frac{d\vec{b}}{ds} = -\gamma\vec{n}, \ \frac{d\vec{n}}{ds} = \gamma\vec{b} - k\vec{t}\right]$$

$$\therefore \frac{d^4\vec{r}}{ds^4} = \left(\frac{d^2k}{ds^2} - k\gamma^2 - k^3\right)\vec{n} + \left(\gamma\frac{dk}{ds} + \gamma\frac{dk}{ds} + k\frac{d\gamma}{ds}\right)\vec{b}$$

$$- 3k\frac{dk}{ds}\vec{t}. \quad \text{---(3)}$$

Taking cross product of (1) and (2) we have

$$\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3} = k^2\gamma\vec{t} + k^3\vec{b} \quad \text{---(4)} \quad \left[\because \vec{n}\times\vec{n} = 0, \ \vec{n}\times\vec{b} = \vec{t}, \ \vec{n}\times\vec{t} = -\vec{b}\right]$$

Taking dot product of (3) and (4) we have

Now,

$$\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3} \cdot \frac{d^4\vec{r}}{ds^4} = (k^2\gamma\vec{t} + k^3\vec{b}) \cdot [\text{expression in (3)}]$$

$$= -3k^3\gamma\frac{dk}{ds} + k^3\left(2\gamma\frac{dk}{ds} + k\frac{d\gamma}{ds}\right).$$

$$\Rightarrow \left[\frac{d^2\vec{r}}{ds^2} \quad \frac{d^3\vec{r}}{ds^3} \quad \frac{d^4\vec{r}}{ds^4} \right] = k^3 \left(k \frac{d\tau}{ds} - \tau \frac{dk}{ds} \right)$$

$$= k^5 \frac{d}{ds}\left(\frac{\tau}{k}\right) \quad —(5) .$$

Since for circular helix, $\frac{\tau}{k}$ = constant

$$\therefore \frac{d}{ds}\left(\frac{\tau}{k}\right) = 0$$

$\therefore$ From (5), $\left[\frac{d^2\vec{r}}{ds^2} \quad \frac{d^3\vec{r}}{ds^3} \quad \frac{d^4\vec{r}}{ds^4} \right] = 0$.

Conversely, let $\left[\frac{d^2\vec{r}}{ds^2} \quad \frac{d^3\vec{r}}{ds^3} \quad \frac{d^4\vec{r}}{ds^4} \right] = 0$

$$\Rightarrow k^5 \frac{d}{ds}\left[\frac{\tau}{k}\right] = 0 \qquad [\text{From } (5)]$$

$$\Rightarrow \frac{\tau}{k} = \text{constant}$$

$$\Rightarrow r = \vec{r}(s) \text{ is a helix}$$

$\boxed{\text{Ex-14}}$: Show that for a plane curve torsion $\tau = 0$ V.H-2012

Ans: Let C is a plane curve i.e. there is a plane that contains C.

Let C lies in the plane $ax + by + cz + d = 0$.

We have $\vec{t} = \frac{d\vec{r}}{ds}$ and $\frac{\vec{n}}{\rho} = \frac{d^2\vec{r}}{ds^2}$

Here $\vec{t}$ and $\vec{n}$ both lies on this plane.

Then this plane is the osculating plane of the curve C.

So, the vector $\vec{b}$ (binormal vector) is orthogonal to this plane at every point on C.

Since $\vec{\alpha} = (a, b, c)$ is vector perpendicular to the plane.

Then the vector $\vec{b}$ at every point is either $\frac{\vec{\alpha}}{|\vec{\alpha}|}$ or $-\frac{\vec{\alpha}}{|\vec{\alpha}|}$.

In both the cases $\vec{b}$ is a constant vector,

$\therefore \quad \frac{d\vec{b}}{ds} = 0 \quad (1)$

We have by Frenet Serret formula $\frac{d\vec{b}}{ds} = -\tau\vec{b}$ [To be proved]

$\therefore$ using (1) we have $\tau = 0$.

$\Rightarrow$ torsion is zero.

Hence the result.

$\boxed{\text{Ex-14.1}}$: If $\tau = 0$ then show that the curve is plane curve.

Ans: If torsion $\tau = 0$ then by Frenet Serret formula,

$$\frac{d\vec{b}}{ds} = 0 \quad \left[\because \frac{d\vec{b}}{ds} = -\tau\vec{b}\right]$$

$\Rightarrow \vec{b}$ is a constant vector.

Let the curve be given by $\vec{r} = \vec{r}(s)$

Then $\frac{d}{ds}(\vec{r}\cdot\vec{b}) = \frac{d\vec{r}}{ds}\cdot\vec{b} + \vec{r}\cdot\frac{d\vec{b}}{ds}$

$= \vec{t}\cdot\vec{b} + \vec{r}\cdot 0$

$= 0 \qquad \left[\frac{d\vec{r}}{ds} = \vec{t} \text{ and } \vec{t}\cdot\vec{b} = 0\right]$

$\Rightarrow \frac{d}{ds}(\vec{r}\cdot\vec{b}) = 0.$

$\Rightarrow \vec{r}\cdot\vec{b}$ is a constant.

Thus, the vectors from the origin to the points on the curve $r(s)$ are all orthogonal to the fixed direction $\vec{b}$.

Hence the curve is a plane curve.

$\boxed{\text{Ex-15}}$: Find the equations of the osculating plane, normal plane and rectifying plane to the twisted cubic

$$x = 2t, \quad y = t^2, \quad z = \frac{t^3}{3} \quad \text{at } t = 1.$$

Ans: We know that the equation of the osculating plane, normal plane and Rectifying plane are

$$(\vec{R}-\vec{r})\cdot\dot{\vec{r}}\times\ddot{\vec{r}}=0 \quad [\text{osculating plane}]$$
$$(\vec{R}-\vec{r})\cdot\dot{\vec{r}}=0 \quad [\text{Normal plane}]$$
$$\& \quad (\vec{R}-\vec{r})\cdot(\dot{\vec{r}}\times\ddot{\vec{r}})\times\dot{\vec{r}}=0 \quad [\text{Rectifying plane}] \text{ respectively.}$$

We have
$$\vec{r}=2t\hat{i}+t^2\hat{j}+\frac{t^3}{3}\hat{k}$$
$$\dot{\vec{r}}=2\hat{i}+2t\hat{j}+t^2\hat{k}$$
$$\ddot{\vec{r}}=2\hat{j}+2t\hat{k}.$$

At $t=1$,
$$\vec{r}=2\hat{i}+\hat{j}+\frac{1}{3}\hat{k}$$
$$\dot{\vec{r}}=2\hat{i}+2\hat{j}+\hat{k}$$
and
$$\ddot{\vec{r}}=2\hat{j}+2\hat{k}.$$

Now,
$$\dot{\vec{r}}\times\ddot{\vec{r}}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&2&1\\0&2&2\end{vmatrix}=2\hat{i}-4\hat{j}+4\hat{k}.$$

Also,
$$(\dot{\vec{r}}\times\ddot{\vec{r}})\times\dot{\vec{r}}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&-4&4\\2&2&1\end{vmatrix}=-12\hat{i}+6\hat{j}+12\hat{k}$$

$\therefore$ Let $\vec{R}=x\hat{i}+y\hat{j}+z\hat{k}$.

$\therefore$ The equation of the osculating plane is

$$(\vec{R}-\vec{r})\cdot\dot{\vec{r}}\times\ddot{\vec{r}}=0$$
$$\Rightarrow \left[(x\hat{i}+y\hat{j}+z\hat{k})-(2\hat{i}+\hat{j}+\tfrac{1}{3}\hat{k})\right]\cdot(2\hat{i}-4\hat{j}+4\hat{k})=0$$
$$\Rightarrow 2(x-2)+(y-1)(-4)+(z-\tfrac{1}{3})4=0$$
$$\Rightarrow x-2y+2z=\frac{2}{3}.$$

The equation of normal plane is
$$(\vec{R}-\vec{r})\cdot\dot{\vec{r}}=0$$
$$\Rightarrow \left[(x\hat{i}+y\hat{j}+z\hat{k})-(2\hat{i}+\hat{j}+\tfrac{1}{3}\hat{k})\right]\cdot(2\hat{i}+2\hat{j}+\hat{k})=0$$
$$\Rightarrow 2(x-2)+2(y-1)+(z-\tfrac{1}{3})\cdot 1=0$$
$$\Rightarrow 2x+2y+z=\frac{19}{3}.$$

The equation of rectifying plane is

$$(\vec{R}-\vec{r})\cdot(\dot{\vec{r}}\times\ddot{\vec{r}})\times\dot{\vec{r}}=0$$

$$\Rightarrow \left[(x\hat{i}+y\hat{j}+z\hat{k})-(2\hat{i}+\hat{j}+\tfrac{1}{3}\hat{k})\right]\cdot(-12\hat{i}+6\hat{j}+12\hat{k})=0$$

$$\Rightarrow -12(x-2)+6(y-1)+12(z-\tfrac{1}{3})=0$$

$$\Rightarrow 6x-3y-6z=7.$$

$\boxed{\text{Ex-16}}$: If $\frac{d\vec{a}}{dt}=\vec{r}\times\vec{a}$ and $\frac{d\vec{b}}{dt}=\vec{r}\times\vec{b}$ then show that

$$\frac{d}{dt}(\vec{a}\times\vec{b})=\vec{r}\times(\vec{a}\times\vec{b}).$$

Ans:

$$\frac{d}{dt}(\vec{a}\times\vec{b})=\vec{a}\times\frac{d\vec{b}}{dt}+\frac{d\vec{a}}{dt}\times\vec{b}$$

$$=\vec{a}\times(\vec{r}\times\vec{b})+(\vec{r}\times\vec{a})\times\vec{b}\quad\left[\because \frac{d\vec{a}}{dt}=\vec{r}\times\vec{a},\ \frac{d\vec{b}}{dt}=\vec{r}\times\vec{b}\right]$$

$$=\vec{r}(\vec{a}\cdot\vec{b})-\vec{b}(\vec{a}\cdot\vec{r})$$

$$-\vec{r}(\vec{a}\cdot\vec{b})+\vec{a}(\vec{r}\cdot\vec{b})$$

$$=\vec{a}(\vec{r}\cdot\vec{b})-\vec{b}(\vec{a}\cdot\vec{r})$$

$$=\vec{r}\times(\vec{a}\times\vec{b})\quad\text{(Proved)}.$$

$\boxed{\text{Ex-16.1}}$: (a) Find the unit tangent vector to any point on the curve $x=t^2+1,\ y=4t-3,\ z=2t^2-6t$.

(b) determine the unit tangent at the point where $t=2$.

Ans: (a) The position vector for any point on the curve is

$$\vec{r}=(t^2+1)\hat{i}+(4t-3)\hat{j}+(2t^2-6t)\hat{k}.$$

$\therefore$ The unit tangent vector is $\vec{t}=\frac{d\vec{r}}{ds}=\frac{d\vec{r}}{dt}\frac{dt}{ds}$

$$=\frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}\quad\left[\because \frac{ds}{dt}=\left|\frac{d\vec{r}}{dt}\right|\right]$$

$$\therefore \frac{d\vec{r}}{dt}=2t\hat{i}+4\hat{j}+(4t-6)\hat{k}.$$

Also, $\left|\frac{d\vec{r}}{dt}\right| = \left|2t\hat{i}+4\hat{j}+(4t-6)\hat{k}\right|$

$$= \sqrt{4t^2+16+(4t-6)^2}$$

$\therefore$ The required unit tangent vector is $\vec{t} = \dfrac{2t\hat{i}+4\hat{j}+(4t-6)\hat{k}}{\sqrt{4t^2+16+(4t-6)^2}}$

(b). At $t=2$, $\vec{t} = \dfrac{4\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{16+16+4}}$

$$= \frac{2}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{1}{3}\hat{k}$$

$\boxed{\text{Ex-17}}$: A particle moves along the curve $x=2t^2$, $y=t^2-4t$, $z=3t-5$ where t is the time. Find the components of velocity and acceleration at time $t=1$ in the direction $\hat{i}-3\hat{j}+2\hat{k}$.

Ans: The position vector of any point on the curve is

$$\vec{r} = 2t^2\hat{i}+(t^2-4t)\hat{j}+(3t-5)\hat{k}.$$

$\therefore$ Velocity $= \dfrac{d\vec{r}}{dt} = 4t\hat{i}+(2t-4)\hat{j}+3\hat{k}.$

$$= 4\hat{i}-2\hat{j}+3\hat{k} \text{ at } t=1.$$

Unit vector in direction $\hat{i}-3\hat{j}+2\hat{k}$ is $\dfrac{\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{1+9+4}}$

$$= \frac{\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{14}} = \hat{a} \text{ (say)}$$

The component of the velocity in the given direction is

$$\frac{d\vec{r}}{dt}\cdot\hat{a}$$

$$= (4\hat{i}-2\hat{j}+3\hat{k})\cdot\left(\frac{\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{14}}\right)$$

$$= \frac{16}{\sqrt{14}}$$

Acceleration $= \dfrac{d^2\vec{r}}{dt^2} = 4\hat{i}+2\hat{j}+0\cdot\hat{k}$. The component of acceleration in the given direction is $(4\hat{i}+2\hat{j})\cdot\left(\dfrac{\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{14}}\right) = -\dfrac{2}{\sqrt{14}}$.

Ex-18 : A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$ where t is the time.

(a) determine its velocity and acceleration at any time

(b) Find the magnitudes of the velocity and acceleration at $t=0$,

Ans: (a) The position vector $\vec{r}$ of the particle is

$$\vec{r} = e^{-t}\hat{i} + 2\cos 3t\,\hat{j} + 2\sin 3t\,\hat{k}.$$

The velocity is $\vec{v} = \frac{d\vec{r}}{dt} = -e^{-t}\hat{i} - 6\sin 3t\,\hat{j} + 6\cos 3t\,\hat{k}$.

and acceleration is $\vec{a} = \frac{d^2\vec{r}}{dt^2} = e^{-t}\hat{i} - 18\cos 3t\,\hat{j} - 18\sin 3t\,\hat{k}$.

(b) At $t=0$ $\vec{v} = -\hat{i} + 6\hat{k}$

$\vec{a} = \hat{i} - 18\hat{j}$.

$\therefore$ The magnitude of $\vec{v}$ at $t=0$ is $\sqrt{(-1)^2 + 6^2} = \sqrt{37}$.

magnitude of acceleration at $t=0$ is $\sqrt{1+18^2} = \sqrt{325}$

Ex-19 : A particle moves so that its position vector is given by $\vec{r} = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$ where ω is constant. Show that

(a) the velocity $\vec{v}$ of the particle is $\perp^r$ to $\vec{r}$.

(b) the acceleration $\vec{a}$ is directed towards the origin and has magnitude proportional to the distance from the origin.

(c) $\vec{r} \times \vec{v} =$ a constant vector.

Ans : (a) $\vec{v} = \frac{d\vec{r}}{dt} = -\omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j}$

Then $\vec{r}\cdot\vec{v} = (\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j})\cdot(-\omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j})$

$= -\omega\sin\omega t\cos\omega t + \omega\sin\omega t\cos\omega t = 0$

$\therefore$ $\vec{v}$ and $\vec{r}$ are perpendicular.

(b) $\frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt} = -\omega^2\cos\omega t\,\hat{i} - \omega^2\sin\omega t\,\hat{j}$

$= -\omega^2\vec{r}$

Then the acceleration is opposite to the direction of $\vec{r}$ i.e. it is directed towards origin. Its magnitude is proportional to $|\vec{r}|$ which is the distance from the origin.

(c)

$$\vec{r}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\omega t & \sin\omega t & 0 \\ -\omega\sin\omega t & \omega\cos\omega t & 0 \end{vmatrix} = \omega\left(\cos^2\omega t + \sin^2\omega t\right)\hat{k}$$

$$= \omega\hat{k}, \text{ a constant vector.}$$

Physically, the motion is that of a particle moving on the circumference of a circle with constant angular speed ω.

The acceleration, directed towards the centre of the circle, is the centripetal acceleration.

$\boxed{\text{Ex-20}}$: Show that the acceleration $\vec{a}$ of a particle which travells along a space curve with velocity $\vec{v}$ is given by

$$\vec{a} = \frac{dv}{dt}\vec{t} + \frac{v^2}{\rho}\vec{n}$$

Where $\vec{t}$ is the unit tangent vector to the space curve, $\vec{n}$ is the unit principal normal and ρ is the radius of curvature.

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Ans: Velocity $\vec{v}$ = magnitude of $\vec{v}$ multiplied by unit tangent vector $\vec{t}$.

$$\therefore \vec{v} = v\vec{t}$$

Differentiate with w.r.t. to t we have

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\vec{t} + v\frac{d\vec{t}}{dt} \quad \text{---(1)}.$$

Now,
$$\frac{d\vec{t}}{dt} = \frac{d\vec{t}}{ds}\cdot\frac{ds}{dt} = k\vec{n}\frac{ds}{dt} \quad \left[\text{By Frenet Serret formula, } \frac{d\vec{t}}{ds} = k\vec{n}\right]$$

$$= \frac{\vec{n}}{\rho}v \quad \left[\frac{ds}{dt} = v\right]$$

---(2).

Using (2) in (1) we have

$$\vec{a} = \frac{dv}{dt}\vec{t} + \frac{v^2}{\rho}\vec{n}$$

This shows that the component of the acceleration is $\frac{dv}{dt}$ in a direction tangent to the path and $\frac{v^2}{\rho}$ in a direction of the principal normal to the path.

$\boxed{\text{Ex-21}}$: If $\vec{r}$ is the position vector of a particle of mass m relative to point O and $\vec{F}$ is the external force on the particle, then $\vec{r}\times\vec{F} = \vec{M}$ is the torque or moment of $\vec{F}$ about O. Show that $\vec{M} = \frac{d\vec{H}}{dt}$ where $\vec{H} = \vec{r}\times m\vec{v}$ and $\vec{v}$ is the velocity of the particle.

Ans : By Newton's law, $\vec{F} = \frac{d}{dt}(m\vec{v})$.

$$\therefore \quad \vec{M} = \vec{r}\times\vec{F} = \vec{r}\times\frac{d}{dt}(m\vec{v})$$

But $\frac{d}{dt}(\vec{r}\times m\vec{v}) = \vec{r}\times\frac{d}{dt}(m\vec{v}) + \frac{d\vec{r}}{dt}\times m\vec{v}$

$$= \vec{r}\times\frac{d}{dt}(m\vec{v}) + \vec{v}\times m\vec{v}$$

$$= \vec{r}\times\frac{d}{dt}(m\vec{v}), \quad \vec{v}\times\vec{v} = 0$$

$$\therefore \quad \vec{M} = \frac{d}{dt}(\vec{r}\times m\vec{v})$$

$$= \frac{d\vec{H}}{dt} \quad \text{where } \vec{H} = \vec{r}\times m\vec{v}.$$

Here $\vec{H}$ is called the angular momentum.

$\boxed{\text{Ex-22}}$: The position vector of a point on the space curve is given by $\vec{r} = t\hat{i} + t^2\hat{j} + \frac{2}{3}t^3\hat{k}$.

Show that the radius of torsion $= \frac{1}{2}(1+2t^2)^2 =$ radius of curvature.

Ans : The position vector is $\vec{r} = t\hat{i} + t^2\hat{j} + \frac{2}{3}t^3\hat{k}$.

$$\therefore \quad \frac{d\vec{r}}{dt} = \hat{i} + 2t\,\hat{j} + 2t^2\hat{k}.$$

$$\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{1+4t^2+(2t^2)^2} = 1+2t^2 \quad \text{—(1)}$$

$$\therefore \quad \vec{t} = \frac{d\vec{r}}{ds} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{\hat{i}+2t\hat{j}+2t^2\hat{k}}{(2t^2+1)} \quad \text{—(2)}$$

$$\therefore \quad \frac{d\vec{t}}{dt} = \frac{(2\hat{j}+4t\hat{k})(2t^2+1) - (\hat{i}+2t\hat{j}+2t^2\hat{k})(4t)}{(1+2t^2)^2}$$

$$= \frac{-4t\hat{i} + (2-4t^2)\hat{j} + 4t\hat{k}}{(1+2t^2)^2}$$

$$\therefore \quad \frac{d\vec{t}}{ds} = \frac{d\vec{t}}{dt}\cdot\frac{dt}{ds}$$

$$= \frac{-4t\hat{i} + (2-4t^2)\hat{j} + 4t\hat{k}}{(1+2t^2)^2}\cdot\frac{1}{(1+2t^2)}$$

$\therefore$ We have by Frenet Serret formulas,

$$\frac{d\vec{t}}{ds} = \kappa\vec{n} \quad \text{—(3)}$$

$$\Rightarrow \kappa = \left|\frac{d\vec{t}}{ds}\right| = \frac{\sqrt{(4t)^2+(2-4t^2)^2+(4t)^2}}{(1+2t^2)^3}$$

$$= \frac{\sqrt{\cancel{16}t^2+4-\cancel{16}t^2+(4t^2)^2+16t^2}}{(1+2t^2)^3}$$

$$= \frac{2}{(1+2t^2)^2} \qquad \left[\because \kappa>0 \ \& \ |\vec{n}|=1\right]$$

$\therefore$ Radius of curvature $\rho = \frac{1}{\kappa} = \frac{(1+2t^2)^2}{2}$

From (3), $\vec{n} = \frac{1}{\kappa}\frac{d\vec{t}}{ds}$

$$= \frac{-4t\hat{i} + (2-4t^2)\hat{j} + 4t\hat{k}}{(1+2t^2)^3} \times \frac{(1+2t^2)^2}{2}$$

$$= \frac{-2t\hat{i} + (1-2t^2)\hat{j} + 2t\hat{k}}{(1+2t^2)}.$$

$$\therefore \quad \vec{b} = \vec{t} \times \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{1+2t^2} & \frac{2t}{1+2t^2} & \frac{2t^2}{1+2t^2} \\ \frac{-2t}{1+2t^2} & \frac{1-2t^2}{1+2t^2} & \frac{2t}{1+2t^2} \end{vmatrix} = \frac{2t^2\hat{i} - 2t\hat{j} + \hat{k}}{1+2t^2}$$

Now,
$$\frac{d\vec{b}}{ds} = \frac{d\vec{b}}{dt}\,\frac{dt}{ds}$$
$$= \frac{4t\hat{i} + (4t^2-2)\hat{j} - 4t\hat{k}}{(1+2t^2)^2} \cdot \frac{1}{(1+2t^2)}$$
$$= \frac{4t\hat{i} + (4t^2-2)\hat{j} - 4t\hat{k}}{(1+2t^2)^3}$$

By Frenet Serret formula,
$$\frac{d\vec{b}}{ds} = -\tau\,\vec{n}$$
$$\Rightarrow \left|\frac{d\vec{b}}{ds}\right| = \tau\,|\vec{n}| \quad \text{as } \tau > 0 .$$

$$\tau = \left| \frac{4t\hat{i} + (4t^2-2)\hat{j} - 4t\hat{k}}{(1+2t^2)^3} \right| \qquad [\because |\vec{n}| = 1]$$
$$= \frac{\sqrt{(4t)^2 + (4t^2-2)^2 + (-4t)^2}}{(1+2t^2)^3}$$

$$\therefore \tau = \text{Torsion} = \frac{2}{(1+2t^2)^2}$$

$\therefore$ Radius of torsion is $\frac{1}{2}(1+2t^2)^2$.

$\therefore$ Radius of torsion $= \frac{1}{2}(1+2t^2)^2 =$ radius of curvature.

Hence the result.

Article: Find the normal and tangent plane to the surface $\vec{r} = \vec{r}(u,v)$ where u and v are two parameters

Let us consider the equation of the surface is
$$\vec{r} = \vec{r}(u,v) . \quad \text{—(1)}$$

Let us consider the point p on the surface S whose curvilinear co-ordinates are (u_0, v_0).

If $v = v_0$, then $\frac{\partial r}{\partial u}$ represents a vector tangent to this curve at the point $P(u_0, v_0)$. Similarly, $\frac{\partial r}{\partial v}$ represents a vector tangent to the curve $u = u_0$ at the point P.

The vector normal to the surface S at the point P is given by $\boxed{\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}}$ ——— (2)

The unit normal vector is $\dfrac{\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}}{\left|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right|}$

The tangent at the point P to every curve on the surface through P lies on the plane through P normal to the vector ②.

The locus of the tangents at P to different curves through P, whose position vector is R_0, is the plane

$$\boxed{(R - R_0) \cdot \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = 0}$$

Where R is the position vector of any point on the plane.

This is called the tangent plane.

$\boxed{\text{Ex} - 23}$: Determine the unit normal to the surface

$$\vec{r} = a\cos u \sin v \,\hat{i} + a \sin u \sin v \,\hat{j} + a \cos v \,\hat{k}, \quad a > 0$$

Ans: The unit normal vector to the given surface is

$$\frac{\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}}{\left|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right|}.$$

$$\therefore \frac{\partial \vec{r}}{\partial u} = -a \sin u \sin v \,\hat{i} + a \cos u \sin v \,\hat{j}$$

$$\frac{\partial r}{\partial v} = a\cos u \cos v \,\hat{i} + a \sin u \cos v \,\hat{j} - a \sin v \,\hat{k}.$$

$$\frac{\partial \vec r}{\partial u} \times \frac{\partial \vec r}{\partial v} = \begin{vmatrix} \hat i & \hat j & \hat k \\ -a\sin u \sin v & a\cos u \sin v & 0 \\ a\cos u \cos v & a \sin u \cos v & -a\sin v \end{vmatrix}$$

$$= -a^2 \cos u \sin^2 v\, \hat i - a^2 \sin u \sin^2 v\, \hat j - a^2 \sin v \cos v\, \hat k$$

$$\therefore \left| \frac{d\vec r}{\partial u} \times \frac{\partial \vec r}{\partial v} \right| = \sqrt{a^4 \cos^2 u \sin^4 v + a^4 \sin^2 u \sin^4 v + a^4 \sin^2 v \cos^2 v}$$

$$= a^2 \sqrt{\sin^4 v + \sin^2 v \cos^2 v}$$

$$= a^2 \sqrt{\sin^2 v} \qquad = a^2 \sin v \text{ if } \sin v > 0$$

$$= -a^2 \sin v \text{ if } \sin v < 0 .$$

$\therefore$ The unit normals given by

$$\pm \left(\cos u \sin v\, \hat i + \sin u \sin v\, \hat j + \cos v\, \hat k \right)$$

Therefore, the unit normal is $\cos u \sin v\, \hat i + \sin u \sin v\, \hat j + \cos v\, \hat k$ which is drawn outward to the surface at (u, v).

Home work :

(i) Show that the equation of the osculating plane, normal plane and the rectifying plane to the curve $x = 3t - t^3$, $y = 3t^2$, $z = 3t + t^3$ at $t = 1$ are $y - z + 1 = 0$, $y + z = 7$ and $x = 2$ respectively.

(ii) For the curve $x = \tan^{-1}(s)$, $y = \frac{\sqrt 2}{2} \log(s^2+1)$, $z = s - \tan^{-1}(s)$ show that (i) $\rho = \frac{1}{\sqrt 2}(s^2+1)$, (ii) $\tau = \frac{\sqrt 2}{s^2+1}$.

(iii) Find the torsion of the curve $x = \frac{2t+1}{t-1}$, $y = \frac{t^2}{t-1}$, $z = t+2$.

[Ans: $\tau = 0$, The curve lies on the plane $x - 3y + 3z = 5$].

(iv) For the curve $\vec r = (2a\cos t, 2a \sin t, bt^2)$ show that $[\dot{\vec r}\ \ddot{\vec r}\ \dddot{\vec r}] = 8a^2 bt$

(v) Show that the curvature of the curve $\vec r = (a\cos 2t, a \sin 2t, 2a \sin t)$ at $(a, 0, 0)$ is $\frac{1}{2a}$

(vii) Find $\vec{t}, \vec{n}, \vec{b}$ for the curve given by

(i) $\vec{r} = (e^t, e^{-t}, \sqrt{2}t)$ at $t=0$

(ii) $\vec{r} = (e^t \cos t, e^t \sin t, e^t)$ at $t=0$.

Show that in (i) the curvature is equal to $\frac{1}{2\sqrt{2}}$

(Ex) Prove that acceleration vector of a particle moving along a space always lies in the osculating plane.

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Ans: At 1st we have ~~have~~ to prove that the acceleration $\vec{a}$ of a particle moving along a space curve with velocity $\vec{v}$ is given by $\vec{a} = \frac{d\vec{v}}{dt}\vec{t} + \frac{v^2}{\rho}\vec{n}$ (have to be proved)

Where $\vec{t}$ is the unit tangent vector to the space curve, $\vec{n}$ is the unit principal normal and ρ is the radius of curvature.

The components of acceleration along the tangent and pricipal normal, and binormal are $\frac{d\vec{v}}{dt}$, $\frac{v^2}{\rho}$, 0.

Therefore, there is no acceleration along the binormal.

So, the acceleration vector at a point lies on the osculating plane.

Ex: If the tangent and the binormal at a point of a curve make angles θ and φ respectively with a fixed direction, show that
$$\frac{\sin\theta}{\sin\varphi} \cdot \frac{d\theta}{d\varphi} = -\frac{\kappa}{\tau}.$$ V. H.

Ans : Let the tangent $\vec{t}$ and binormal $\vec{b}$ at a point of a curve make angles θ and φ with the fixed direction, say $\vec{a}$ in space. Then

$$\vec{t}\cdot\vec{a} = |\vec{t}|\,|\vec{a}|\cos\theta$$
$$= |\vec{a}|\cos\theta \quad \text{--(1)}$$

and
$$\vec{b}\cdot\vec{a} = |\vec{b}|\,|\vec{a}|\cos\varphi$$
$$= |\vec{a}|\cos\varphi \quad \text{--(2)}$$

Diff. (1) & (2) w.r. to s we have

$$\frac{d\vec{t}}{ds}\cdot\vec{a} = -|\vec{a}|\sin\theta\,\frac{d\theta}{ds}$$
$$\Rightarrow \kappa\vec{n}\cdot\vec{a} = -|\vec{a}|\sin\theta\,\frac{d\theta}{ds} \quad \text{--(3)} \qquad \left[\because \frac{d\vec{t}}{ds} = \kappa\vec{n}\right]$$

and
$$\frac{d\vec{b}}{ds}\cdot\vec{a} = -a\sin\varphi\,\frac{d\varphi}{ds}$$
$$\Rightarrow -\tau\vec{n}\cdot\vec{a} = -a\sin\varphi\,\frac{d\varphi}{ds} \quad \text{--(4)} \qquad \left[\frac{d\vec{b}}{ds} = -\tau\vec{n}\right]$$

Dividing (3) by (4) we have

$$\frac{\sin\theta}{\sin\varphi}\,\frac{d\theta}{d\varphi} = -\frac{\kappa}{\tau}.$$