Solution of Algebric and Trancendendal Equation.

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Solution of Algebric and Transcendental Eqn.

Introduction :-

An equation $f(x)=0$ is said to be algebric or transcendental according as $f(x)$ is purely a polynomial in x or $f(x)$ contains some other functions namely trigonometric, logarithmic, exponential etc.

Thus the equations like $5x^7+3x^2+7x+7=0$, $x^3-8x^2+3=0$ etc are the examples of algebric equations. On the other hand $e^x+3\log_e x+\cos x=0$, $xe^x+5\cos(e^x)+\sin x=0$ etc are the examples of transcendental equations.

We know some methods to solve the algebric equations. As Cardan's method gives the solution of Cubic equations. For the solution of biquadratic equations, we may use Euler's method, Ferrari's method etc in algebra.

In general, there is no closed form formula to evaluate the roots of algebric equation of degree greater than two.

The algebric methods fail to compute the roots of the transcendental equations like $e^x+3\log_e x+\cot x=0$, $x^x+e^x\ln x+\sin x=0$. In such cases.

We shall discuss the numerical methods for computing the roots of an equation algebric or transcendental of the form $f(x)=0$

We assume that

(i) the function $f(x)$ is continuous and continuously differentiable upto any number of times.

(ii) $f(x)=0$ has no multiple root.

Each & every numerical method, consists of two parts

(a) Find out the location of roots (crude approximation)

(b) To improve the rough value of each root to any desired degree of accuracy.

Here the methods discussed are based on iterative techniques. We have to keep in mind that considering iteratives techniques

(a) the convergence or divergence of the iterative techniques.

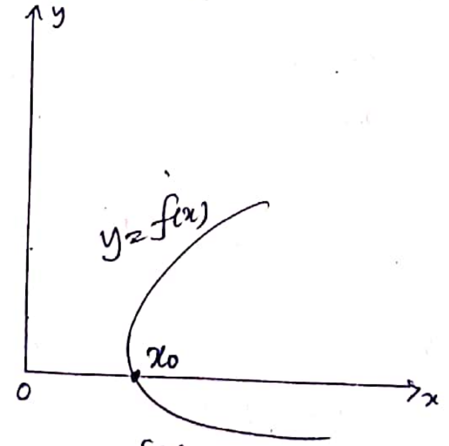
(b) Consider the rate of convergence.

Method for Finding Crude (initial) Approximation to a Real Root:

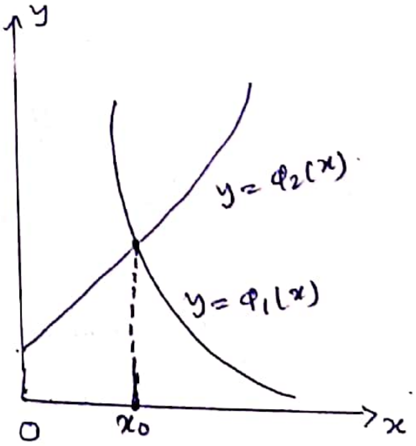
Here two methods are used for finding real root of a numerical equation. (i) Graphical Method (ii) Analytical Method.

Graphical Method: (a) If $f(x)$ is simple, we shall draw the graph of $y=f(x)$ with respect to the rectangular axes $x'ox$ & $y'oy$. Then the points at which the graph meets x-axis are the location of the roots of the equation $f(x)=0$ by putting y-coordinate zero.

(b) If $f(x)$ is not simple, rather complicated in form, we re-write the equation $f(x)=0$ as $\varphi_1(x) = \varphi_2(x)$ where $\varphi_1(x)$ & $\varphi_2(x)$ are simple functions such that we can draw conveniently the graphs $y=\varphi_1(x)$ & $y=\varphi_2(x)$ with respect to rectangular axes. Then x-coordinate of the point of intersection of the graphs give the crude approximation of the real root of the equation $f(x)=0$.



(a)



(b)

Analytical Method: If a function $y=f(x)$ is real valued and contin[uous] in the closed interval $[a, b]$ & if $f(a)$ & $f(b)$ are of opposite sign then there are at least one real root α of the equation $f(x)=0$ between a & b. i.e $a<\alpha<b$.

Bisection Method : C.H - '96, 98, 2000

This method based on the theorem which states that "if a function $f(x)$ is continuous in a closed interval $[a, b]$ & $f(a)$ & $f(b)$ are of opposite signs [i.e $f(a)f(b)<0$] then there exist at least one root of $f(x)=0$ between a & b.

Let $f(a)$ & $f(b)$ are of opposite signs i.e $f(a)f(b)<0$. Let $\xi = \frac{a+b}{2}$, the middle point of $[a, b]$. Now if $f(\xi)=0$ then ξ is a root of $f(x)=0$. If $f(\xi) \neq 0$, then either $f(a)f(\xi)<0$ or $f(\xi)f(b)<0$.

If $f(a)f(\xi)<0$, then root will lie between $[a, \xi]$ and if $f(b)f(\xi)<0$ then the root lies between $[\xi, b]$. We thus reduce the interval from $[a, b]$ to $[a, \xi]$ or $[\xi, b]$.

Then as before, the newly reduced interval in which the root lies is again halved and the process is repeated untill the root is obtained to the desired accuracy.

[N.B. → The method of bisection is also called bracketing method since the method successively reduces two ends (brackets) of an interval surrounding the real root.]

Q. What are the advantages & disadvantages of this method; (Bisection)

C.H - 96, 98, 2000

Advantages: In the method of bisection, at any stage of iteration the approximate value of the desired root of the equation $f(x)=0$ does not depends on the value of $f(x_n)$ but on their sign. So this advantage makes the computation at any step very easy. This method is essentially convergent & there is no restriction.

Disadvantages: The accuracy of the process is not satisfactory i.e not good. So the numbers of iterations may have to be repeated to a large numbers of times to get a moderately good result.

Show that Bisection method will sure converge. C.H.-'2000

By graphical method or tabulation method in which $f(a_0)\cdot f(b_0) < 0$,

we determine small interval $[a_0, b_0]$.

a_0 ——— $x_1 = \frac{a_0+b_0}{2}$ ——— b_0

So there is only one real root of $f(x)=0$.

Now $[a_0, b_0]$ be divided in two equal parts by $x_1 = \frac{a_0+b_0}{2}$ & $f(x_1)$ is calculated.

If $f(x_1)=0$ then x_1 is exact root of $f(x)=0$. If $f(x_1)\neq 0$, then either $f(x_0)f(x_1)<0$ or $f(x_1)f(b_0)<0$. For convenience root α lies between $[x_1, b_0]$. We rename the interval as $[a_1, b_1]$ so that $b_1 - a_1 = \frac{1}{2}(b_0 - a_0)$.

Now we take $x_2 = \frac{a_1+b_1}{2}$ & $f(x_2)$ is computed then either $f(a_1)f(x_2)<0$ or $f(x_2)f(b_1)<0$. If $f(x_2)=0$ then x_2 is the exact root of $f(x)=0$.

We assume that $f(a_1)f(x_2)<0$ then the root α of $f(x)=0$ lies in $[a_1, x_2]$. We call it as $[a_2, b_2]$ where $b_2 - a_2 = \frac{1}{2^2}(b_0 - a_0)$. Proceeding in this manner, $x_{n+1} = \frac{a_n+b_n}{2}$ which is $(n+1)$th approximation of the root α of $f(x)=0$ & lies in the interval $[a_n, b_n]$ where $b_n - a_n = \frac{1}{2^n}(b_0 - a_0)$, $a_0 \le a_n < b_n \le b_0$, $\forall n$

If ϵ_{n+1} be the error in approximating α by x_{n+1} then

$$\epsilon_{n+1} = |\alpha - x_{n+1}| < b_n - a_n < \frac{b_0 - a_0}{2^n} \to 0 \text{ as } n\to\infty.$$

$\Rightarrow \lim_{n\to\infty} x_{n+1} = \alpha$.

Thus the iterative process surely converges.

Computation Scheme of Bisection Method:

When $f(a_0)>0$ & $f(b_0)<0$

n	a_n (+ve)	b_n (−ve)	$x_{n+1} = \left(\frac{a_n+b_n}{2}\right)$	$f(x_{n+1})$
0	a_0	b_0	$x_1 = \left(\frac{a_0+b_0}{2}\right)$	$f(x_1)>0$
1	$x_1 = (a_1)$	$b_0 = (b_1)$	$x_2 = \left(\frac{a_1+b_1}{2}\right)$	$f(x_2)>0$
2	$x_2 (= a_2)$	$b_0 (= b_2)$	$x_3 \left(= \frac{a_2+b_2}{2}\right)$	$f(x_3)<0$
3	$a_2 (= a_3)$	$x_3 (= b_3)$	$x_4 = \left(\frac{a_3+b_3}{2}\right)$	$f(x_4)<0$
4	$a_3 (= a_4)$	$x_4 (= b_4)$	$x_5 \left(= \frac{a_4+b_4}{2}\right)$	$f(x_5)>0$

& so on.

Example: Solve the equation $x^3-9x+1=0$ for the root lying between 2 & 3, Correct to 3-significant figures (by Bisection Method) C. H-'87

Soln: Let $f(x)=x^3-9x+1$ Here $f(2)=-9$, $f(3)=1$.

$\therefore f(2)f(3)<0$.

n	a_n (−ve)	b_n (+ve)	$x_{n+1}=\frac{a_n+b_n}{2}$	$f(x_{n+1})$
0	2	3	2.5	−5.8
1	2.5	3	2.75	−2.9
2	2.75	3	2.88	−1.03
3	2.88	3	2.94	−0.05
4	2.94	3	2.97	0.47
5	2.94	2.97	2.955	0.21
6	2.94	2.955	2.9475	0.08
7	2.94	2.9475	2.9438	0.017
8	2.94	2.9438	2.9419	−0.016
9	2.9419	2.9438	2.9428	−0.003

Thus the root of this equation is 2.94 upto three significant figures.

Q. Explain the method of iteration for numerical solution of the equation $y=f(x)=0$. Why this method is called fixed pt. C. H- '99, '97

Ans In this method we can write the eqn $f(x)=0$ in the form $x=\varphi(x)$ — (1)
This method is based on the principle of finding a ~~sequence~~ successive approximation to a root by forming a sequence $\{x_n\}$ which converges to a root ξ such that $\xi=\varphi(\xi)$.

Using graphical or tabulation method we 1st find a location or crude approximation of a real root ξ of $f(x)=0$.

Let $x = x_0$ be an approximate value of the desired root. We then substitute this value of x on the right hand side of (1) & get the 1st approximation as

$$x_1 = \varphi(x_0)$$

Then the successive approximations are calculated as

$$x_2 = \varphi(x_1)$$
$$x_3 = \varphi(x_2)$$
$$\cdots\cdots$$
$$x_{n+1} = \varphi(x_n).$$

The above iteration is generated by the formula $x_{n+1} = \varphi(x_n)$ & is called the iteration formula. Where x_n is the nth approximation of the root ξ of $f(x)=0$. This process is continued untill $|x_{n+1} - x_n| < \epsilon$, ϵ is the error tolerance.

2nd Part:- The root ξ of the given equation satisfies $\xi = \varphi(\xi)$. This iteration is called fixed point as ξ remains fixed under the mapping φ. defined $x = \varphi(x)$.

N.B.:- The intial approximation x_0 does not guarantee that the sequence obtained will converge to a root. If x_0 is near the root, we can expect the convergence but this not always be case.

Sometimes the initial guess just greater than a root may start a divergence sequence where as just less than a root, may form a convergence seque.

Q. Obtain the condition of convergence of fixed point iteration process.

V. H – '91, '95.
C. H – '99, '97.

The presentation of $f(x)=0$ as $x = \varphi(x)$ is not unique, therefore the convergence of $\{x_n\}$ depends upon the nature of $\varphi(x)$.

Assume that $x = \xi$ is a root of the equation $x = \varphi(x)$. Hence we get

$$\xi = \varphi(\xi) \quad \cdots\cdots (1)$$

& ξ lies in the interval $[a,b]$. Let $\varphi(x)$ & $\varphi'(x)$ be continuous in $[a,b]$ & there is a proper fraction l such $|\varphi'(x)| \le l < 1, \ \forall x \in [a,b]$. ---- (2)

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The successive approximations for the iteration process is

$$\left.\begin{aligned} x_1 &= \varphi(x_0) \\ x_2 &= \varphi(x_1) \\ &\cdots\cdots \\ x_{n+1} &= \varphi(x_n). \end{aligned}\right\} \longrightarrow (3)$$

$\therefore$ Now $x_1 = \varphi(x_0)$ & $\xi = \varphi(\xi)$. [from (1)]

Subtracting these we have $(\xi - x_1) = \varphi(\xi) - \varphi(x_0)$

$$= (\xi - x_0)\,\varphi'(\xi_0), \quad x_0 < \xi_0 < \xi.$$

[By M.V.T]

In this manner we obtain

$$(\xi - x_2) = (\xi - x_1)\,\varphi'(\xi_1), \quad x_1 < \xi_1 < \xi.$$

$$(\xi - x_3) = (\xi - x_2)\,\varphi'(\xi_2), \quad x_2 < \xi_2 < \xi$$

$$\cdots\cdots\cdots\cdots$$

$$(\xi - x_{n+1}) = (\xi - x_n)\,\varphi'(\xi_n), \quad x_n < \xi_n < \xi.$$

Thus from these relations we have

$$(\xi - x_{n+1}) = (\xi - x_0)\,\varphi'(\xi_0)\,\varphi'(\xi_1)\cdots\cdots\varphi'(\xi_n).$$

$$\therefore |\xi - x_{n+1}| = |(\xi - x_0)|\,|\varphi'(\xi_0)|\,|\varphi'(\xi_1)|\cdots|\varphi'(\xi_n)|.$$

$$\leq |(\xi - x_0)|\, l^{n+1} \quad \text{[by equation (2)]}$$

$$\therefore \lim_{n\to\infty} |\xi - x_{n+1}| \leq |\xi - x_0| \lim_{n\to\infty} l^{n+1} \to 0 \text{ if } l < 1 \text{ i.e. } |\varphi'(x)| < 1$$

$$\to \infty \text{ if } l > 1 \text{ i.e. } |\varphi'(x)| > 1.$$

$\therefore$ Thus $\lim\limits_{n\to\infty} x_{n+1} = \xi$ iff $|\varphi'(x)| \leq l < 1$ in $[a, b]$.

Hence it follows that the sequence of approximations $x_0, x_1, \cdots\cdots, x_n$ converges to the desired root ξ if $|\varphi'(x)| < 1$.

Q. **What are the advantages and disadvantages of iteration method?** C.H–'96, 2000

Advantages: Iteration method is self corrected i.e. if an error occurs in the computation of an approximation x_n, the incorrect value of x_n may be taken as a new approximation to the desired root and finally we will get the correct root. But if the aforesaid error is very large, the fixed point iteration method may not be convergent.

Disadvantages: (i) The disadvantage of the fixed point iteration method is that it is conditionally convergent i.e. $|\varphi'(x)| < 1$.

(ii) It is some time becomes difficult to express the expression $f(x)=0$ as $x = \varphi(x)$, where $-1 < \varphi'(x) < 1$ in the general interval.

Furthermore this method is not always convergent rapidly.

Geometrical meaning of iteration method.

~~CH~~'98

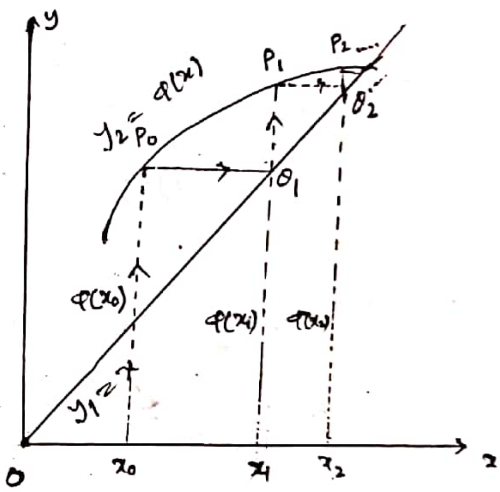
Here the equation of the form $x = \varphi(x)$ & the successive approximations ~~are~~ to the root ~~given~~ by $x_0, x_1, x_2, \ldots, x_n$. Then the relations are:

$x_1 = \varphi(x_0)$

$x_2 = \varphi(x_1)$

$x_{n+1} = \varphi(x_n)$ etc. can be pictured as points by the following geometric construction.

Draw the graph $y_1 = x$ & $y_2 = \varphi(x)$. Since $|\varphi'(x)| < 1$ for convergent, the inclination of the curve $y_2 = \varphi(x)$ must be less than 45^0.



$(0 < \varphi'(x) < 1)$

Now to trace convergence of the iteration process, draw the ordinate $\varphi(x_0)$ at x_0. Then from the point P_0 draw a line ∥ to OX until it intersects the line $y_1 = x$ at the point Q_1. Now Q_1 is the 1st iteration equation $x_1 = \varphi(x_0)$. Then draw Q_1P_1, Q_2P_2, …… etc indicated by arrows. Now the points $Q_1, Q_2, Q_3, \ldots$ thus approach the point of intersection of the curve $y_1 = x$ & $y_2 = \varphi(x)$ as the iteration proceeds.

Example: Find the root of the equation $3x - \cos x - 1 = 0$ by iteration method, correct to four significant figures.

Ans: Let $f(x) = 3x - \cos x - 1$

$\therefore$ $f(0) = -2 < 0$ & $f(1) = 1.45 > 0$.

Thus one root of $f(x)=0$ between 0 & 1.

We rewrite the equation as
$$x = \frac{\cos x + 1}{3} = \varphi(x) \quad \therefore \varphi'(x) = -\frac{\sin x}{3}.$$

$\therefore$ $|\varphi'(x)| < 1$ as $|\sin x| < 1$.

Here we take $x_0 = 0$.

n	x_n	$\varphi(x_n)$
0	0	0.6
1	0.6	0.61
2	0.61	0.606
3	0.606	0.6073
4	0.6073	0.60706
5	0.60706	0.60711
6	0.60711	0.60710

Thus 0.6071 is a root of the equation, correct to four significant figures.

Example: In order to find the real root of $x^3 - x + 1 = 0$ near $x = 1$, which of the following iteration functions give convergent sequences?
(i) $x = x^3 + 1$ (ii) $x = \frac{x+1}{x^2}$ (iii) $x = \sqrt{\frac{x+1}{x}}$.

[Type] – V.H.'99

Soln:- For the form (i) $\varphi(x) = x^3 - 1$, $\varphi'(x) = 3x^2$
Hence $|\varphi'(x)| > 1$, this form would give divergent sequence.

(ii) For the form, $x = \frac{x+1}{x^2}$, $\varphi(x) = \frac{1}{x} + \frac{1}{x^2}$, $\varphi'(x) = -\frac{1}{x^2} - \frac{2}{x^3}$.

$\therefore$ $|\varphi'(1)| = 3 > 1$, this would give divergent sequence.

(iii) For the form $x = \sqrt{\frac{x+1}{x}}$, $\varphi(x) = \sqrt{\frac{x+1}{x}}$, $\varphi'(x) = \frac{1}{2}\left(\frac{x+1}{x}\right)^{-1/2}\left(-\frac{1}{x^2}\right)$

$\therefore$ $\varphi'(x) = \frac{1}{2\sqrt{2}} < 1$, this would give convergent sequence.

For the soln of the eqn $x^3 - x + 1 = 0$ we take $x = \sqrt{\frac{x+1}{x}}$ as $\varphi(x)$, i.e. $\varphi(x) = \sqrt{\frac{x+1}{x}}$.

Q. Describe Newton Raphson method to find a real root of the equation $f(x)=0$ where $f(x)$ is continuous function of x. Give a geometrical interpretation of the method. Comment on the limitations of N-R. method.

V.H-'2000, '96, '03
C.H-'96, '2000, '98.

Ans:

This is also an iterative method & is used to find a simple (non repeated) root of an equation $f(x)=0$. The object of this method is to correct the approximate root x_0 (say) to its desired root α.

Initially a crude approximation, small interval $[a_0, b_0]$ is found out in which only one root α (say) of $f(x)=0$ lies.

Let $x=x_0=a_0$ ($a_0 \le x_0 \le b_0$) is an approximate value of the desired root of the equation $f(x)=0$. & h be the small correction to it.

Then $x_1 = x_0+h$ is the correct root. —(1)

$\therefore$ $f(x_1)=0$

$\Rightarrow f(x_0+h)=0$

$\Rightarrow f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\cdots = 0$ [By Tailors expansion]

Now since h is small, we may neglect the terms of 2nd and higher order of h, we get $f(x_0)+hf'(x_0)=0 \Rightarrow h=-\frac{f(x_0)}{f'(x_0)}$

$\therefore$ $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ [using in (1)]

Further if h_1 be the correction on x_1 then $x_2 = x_1+h_1$ is the correct root of $f(x)=0$ $\Rightarrow$ $f(x_2) = f(x_1+h_1)=0$

$\therefore$ $f(x_1)+h_1f'(x_1)+\frac{h_1^2}{2!}f''(x_1)+\cdots\cdots = 0$

$\Rightarrow f(x_1)+h_1f'(x_1)=0$ [Neglecting 2nd and higher powers of h_1]

$\Rightarrow h_1 = -\frac{f(x_1)}{f'(x_1)}$

$\therefore$ $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

Proceeding in this way we have $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$\cdots\cdots\cdots\cdots$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

The formula generates a sequence of successive approximations to get the correct (desired) root α of $f(x)=0$. provided the sequence is convergent.

This formula is known as the iteration formula for Newton Raphson Methods.

This method is continued until $|x_{n+1}-x_n|<\epsilon$, ϵ is the error tolerance.

Geometrical Interpretation of Newton Raphson : C.H-'98

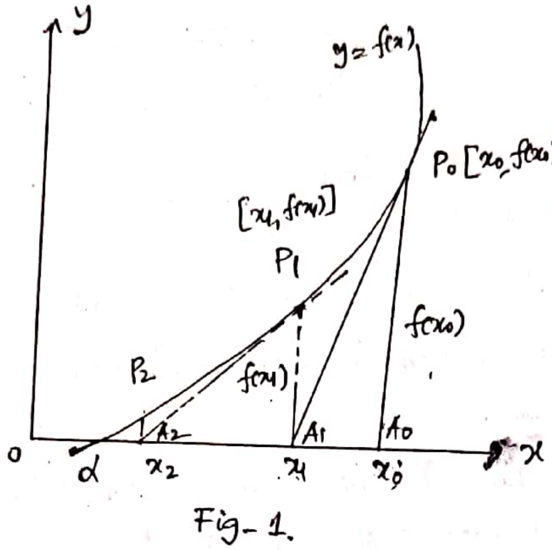


Fig-1.

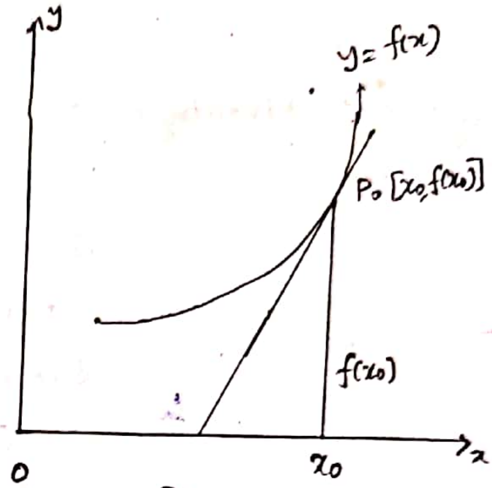


Fig-2.

We draw the graph of the curve $y=f(x)$ w.r.to ox & oy as axes. Let the tangent at $P_0[x_0, f(x_0)]$ meet the x-axis, A_1 where $OA_1 = x_1$. in fig-1. & the tangent at $P_1[x_1, f(x_1)]$ meet the axis at A_2 where $OA_2 = x_2$ etc.

Thus $P_0A_0 = A_1A_0 \tan\angle P_0A_1A_0 = A_1A_0\, f'(x_0)$.

$$\Rightarrow f(x_0) = (x_0 - x_1)\, f'(x_0)$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Thus it is clear that the successive approximations of the root i.e $x_1, x_2, \ldots \ldots x_{n+1}$ are obtained by the points at which the tangents at $x_0, x_1, x_2, \ldots \ldots x_n$ to the curve $y=f(x)$ meet the x-axis.

Limitations: (i) Newton-Raphson method depend on the value of $f'(x)$.

If the value of $f'(x)$ is large in the neighbourhood of the desired root, then small correction will be added to get desired root.

If the value of $f'(x)$ is small then N-R method fails.

(ii) The initial approximation must be taken very close to the desired root otherwise the iterations may diverge.

Q. What are the advantages & disadvantages of Newton-Raphson method
BU '96

Ans: **Advantages:** (i) Newton-Raphson method converges rapidly as the rate of convergence of this method is quadratic.

So if the initial approximation is very close to the root, then the convergence in N-R method is faster than fixed point iteration method.

(ii) Newton Raphson method can be used for finding a complex root if the initial guess x_0 is a complex number.

Disadvantages: (i) N-R method fails when $f'(x)=0$ or every small neighbourhood of the root.

(ii) In this we have compute $f'(x_n)$ & $f(x_n)$ but sometimes the expression $f'(x_n)$ becomes cumbours so computation of $f'(x_n)$ become troublesome.

(iii) The initial approximation must be taken very close to the desired root otherwise the iteration may diverge.

Q. Find the condition of convergence of Newton-Raphson method:-

Ans: As Newton-Raphson method is iterative method then comparing N-R method with fixed point iteration process as

$$\phi(x) = x - \frac{f(x)}{f'(x)}.$$

Thus the above sequence will be convergent iff $|\phi'(x)| < 1$.

$$\therefore |\phi'(x)| = \left|1 - \frac{\{f'(x)\}^2 - f(x)f''(x)}{\{f'(x)\}^2}\right| < 1.$$

$$\text{ie } \left|\frac{f(x)\,f''(x)}{\{f'(x)\}^2}\right| < 1$$

$$\therefore |\{f'(x)\}^2| > |f(x)\,f''(x)|$$ which is the required condition.

Q. What is order of convergence? Prove that order of convergence of iteration method is linear.

Ans: Let x_{n+1} be the $(n+1)$th approximation to a desired root, and ϵ_{n+1} be the corresponding error. Let $\epsilon_{n+1} + \xi = x_{n+1}$.

Let the sequence $\{x_n\}$ of numbers converges to ξ and

$x_{n+1} = \xi + \epsilon_{n+1}$ for $n \geq 1$. If two positive numbers $A \neq 0$ and $p > 0$ exist and $\lim_{n\to\infty} \frac{\epsilon_{n+1}}{\epsilon_n^p} = A$.

Then the sequence is said to converge to ξ with the order of convergence p. A is called the asymptotic error constant.

2nd Part: In iteration method, convergence depends on the suitable choice of the iteration function $\phi(x)$ and x_0, the initial approximate root. In the iteration, $f(x)=0$ can be put as

$x = \phi(x)$, provides $|\phi'(x_0)| < 1$.

Let x_n converges to the desired root ξ so that $\xi = \phi(\xi)$.

Let $\epsilon_{n+1} + \xi = x_{n+1}$. & iteration scheme is $x_{n+1} = \phi(x_n)$.

$$\epsilon_{n+1} = x_{n+1} - \xi$$
$$= \phi(x_n) - \phi(\xi)$$
$$= \phi(\epsilon_n + \xi) - \phi(\xi)$$
$$= \phi(\xi) + \epsilon_n \phi'(\xi) + \frac{\epsilon_n^2}{2!}\phi''(\xi) + \cdots - \phi(\xi)$$
$$= \epsilon_n \phi'(\xi) + O(\epsilon_n^2)$$

$\therefore$ $\epsilon_{n+1} \simeq \epsilon_n \phi'(\xi)$, [Neglecting the squares and higher powers of ϵ_n]

$\therefore \lim_{n\to\infty} \frac{\epsilon_{n+1}}{\epsilon_n} \simeq \phi'(\xi)$. Thus, the order of convergence of iteration method is linear.

Q. Show that the Newton-Raphson method has a quadratic rate of convergence. , V. H - 2001, 90.

Ans: Let x_{n+1} be the $(n+1)$th approximation to the actual root ξ of the equation $f(x)=0$ then $f(\xi)=0$ also, ϵ_{n+1} be the corresponding error, let, $x_{n+1} = \epsilon_{n+1} + \xi$, $x_n = \epsilon_n + \xi$.

The iteration scheme for Newton Raphson method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The relation (1) becomes $\epsilon_{n+1} + \xi = \epsilon_n + \xi - \dfrac{f(\epsilon_n + \xi)}{f'(\epsilon_n + \xi)}$.

$$\Rightarrow \epsilon_{n+1} + \xi = \epsilon_n + \xi - \frac{f(\xi) + \epsilon_n f'(\xi) + \frac{\epsilon_n^2}{2} f''(\xi) + \cdots}{f'(\xi) + \epsilon_n f''(\xi) + \cdots}$$

$$\Rightarrow \epsilon_{n+1} = \epsilon_n - \frac{f'(\xi)\left[\epsilon_n + \frac{\epsilon_n^2}{2}\frac{f''(\xi)}{f'(\xi)} + \cdots\right]}{f'(\xi)\left[1 + \epsilon_n \frac{f''(\xi)}{f'(\xi)} + \cdots\right]}$$ [By Taylor's series] [$\because f(\xi)=0$]

$$= \epsilon_n - \left[\epsilon_n + \frac{\epsilon_n^2}{2}\frac{f''(\xi)}{f'(\xi)} + \cdots\right]\left[1 - \epsilon_n \frac{f''(\xi)}{f'(\xi)} + \cdots\right]$$

$$= \epsilon_n - \epsilon_n - \frac{\epsilon_n^2}{2}\frac{f''(\xi)}{f'(\xi)} + \epsilon_n^2 \frac{f''(\xi)}{f'(\xi)} + O(\epsilon_n^3)$$

$$= \frac{1}{2}\frac{\epsilon_n^2}{}\frac{f''(\xi)}{f'(\xi)} + O(\epsilon_n^3)$$

$$\therefore \epsilon_{n+1} = \frac{\epsilon_n^2}{2}\frac{f''(\xi)}{f'(\xi)}.$$ [Neglecting the terms of order ϵ_n^3 and higher powers]

This shows that the subsequent error at each step is proportional to the square of the previous error.

Thus, it is clear that Newton-Raphson method has a quadratic rate of convergence.