M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

Paper-VI

PART-II

Group-B

Module No. - 110
OPTIMAL CONTROL

Structure :

1. Introduction : Most mathematical models in physical science, mechanics or economics can be characterised by a set of functions $x_1(t), x_2(t), ..., M_x(t)$ which satisfy a set of first order differentical equations of the form

$$\frac{dxi}{dt} = fi(x_i, x_2 ..., x_n; u_1, u_2, ..., un; t) \quad (1)$$

Where fi is a function of $x_1, x_2, ..., x_n; u_1, u_2, ..., un$ and t. The variables $u_1, u_2, ..., un$ are called the control variables and $x_1, x_2, ..., xn$ are called state variables.

Optimal control deals with the problem of finding a control law for such a given system of differential equations describing he paths of the control variables that minimizes a cost functional which is a function of the state variables and control variables.

For a constrained set of equations, there will be some oundary conditons. If to $<t<t_1$ is the time interval under consideration, then x_i's are normally specified at t = to i.e., x (to) are given for i = 1, 2, ..., n. Thus for $u;j$ = 1, 2, ..., n, the above set of equations (1) determines xi's uniquely. Normal control problem is to choose the controls $u_1, u_2, ..., un$ so that the system moves from its unitial state $\{x_1(0), x_2(0), ..., x_n(0)\}$ to some prescribed state at $t - t_1$. If this can be achieved, we say that the system is controllable.

If there exist more than one set of controls which control the system, then our problems is to find the set of controls which control the system in the best way, i.e., we must minimize or maximize the functional of the form

$$J = \int_0^t F(x_1, x_2 ..., x_n, u_1, u_2; t)\, dt$$

We organize this module into the following sections :

2. Performance indices : In modern control theory, the optimal control problem is to find a control which causes the dynamical system to reach a target or follow a state variable (ortrajectory) and at the same time extremize a performance index which may take several forms as described below.

(i) performance index for time-optimal control system"

Here, we try to transfer a system from an arbitrary initial state x_0 at time t_n to a specified final state x_1 at time t_1 in minimum time

The corresponding performance index is $J = \int_{t_0}^{t_1} d_t = t_1 - t_0 = t^*(say)$

(ii) Performance index for fuel optimal control system :

Consider a spacecrat problem. Let $u(t)$ be the thrust of a rocket engine and assume that the magnitude $|u(t)|$ of the thrust is proportional to the rate of fuel consumption. In order to minimize the total expenditure of fuel, we may formulate the performance index as

$$J = \int_{t_0}^{t_1} |u(t)| dt$$

For several controls, we may write it as

$$J = \int_{t_0}^{t_1} \sum_{i=1}^{m} Ri|ui(t)| dt,$$ where R is a weighting function.

(iii) performance index for minimum energy control system :

Consider $ui\ (t)$ as the current in the ith loop of an electric network then $\sum_{i=1}^{m} ui^2(t)$ (Where, ri is the resistance of the i-th loop) is the total power or total rate of energy expenditure of the network, then for minimization of the total expended energy, we have a performance criterion as $J = \int_{t_0}^{t_1} \sum_{i=2}^{m} ui^2(t) r_i dt$

or, in general $J = \int_{t_0}^{t_1} u'(t) R u(t) dt$

Where R is a positive definite matrix and prime (') denotes the transpose.

Similarly, we can think of minimization of the integral of the squared error of a tracking system. We then have $J = \int_{t_0}^{t_1} x'(t) Q(x) t\, dt,$

Where $x_d(t)$ is the desired value, $x_a(t)$ is the actually obtained value and $x(t) = x_a(t) - x_d(t)$, is the error. Here, Q is a weighting matrix which can be positive semi-difinite.

(iv) performance index for terminal control system : In a terminal target problem we are interested in minimizing the error between the desired target position $x_d(ti)$ and the actual target position $xa)ti)$ and the end of the manever or at the final time ti. The terminal (final) error is $x(t1) = uxa(t) - xd(ti)$. Taking care of positive and negative values or error and weghting factors, we construct the cost functin as $J = x'(t_1\ V\ x(t_1)$

Which is also called the terminal cost function. Here V is positive semi definite matrix.

(v) performance index for general optimal control systems "Combining the above formulations, we have a performance index in general form as

$$J = x'(t_1) V x(t_1) + \int_{t_0}^{t_1} [x'(t) Q x)t) + u'(t) R u(t)] dt$$

or, $$J = G(x(t_1), t_1) + \int_{t_0}^{t_1} f(x(t), u(t), t) dt \qquad (2)$$

Where R is a positive definite materix, and Q and V are positive semi-definite matrices, respectively. Note that the matrics Q and R may be time varying. The particular form of performance index given by equation (2) is called a quadratic (in terms of states and controls) form.

The problems arising in optimal controls are classified based on the structure of the performance index J. If the performance index (PI) contains the terminal cost function $G(x(t),...u(t),t)$ only, it is called the Mayer problem, if the PI has only the integrl cost term, it is called the lagrage problem and the problem is of the Bolaz type if the PI contains both the terminal cost term and the integral cost terms.

However, in this coure module, we intend to study the control problems of Lagrange type only.

It is also to be noted that the performance indices are sometimes referred to as payoff functionals.

3. Calculus of variations : To optimize a quantity which appears as an integral, we use the calculus of variations. In other words, calculus of variation can be applied to obtain the necessary condition for a quantity appearing as an integral has either a minimum or a maximum value, i.e., stationary value.

Let us consider the simplest integral.

$$J = \int_{t_0}^{t_1} F(x,\dot{x},t)\,dt, \text{where } \dot{x} = \frac{dx}{dt} \tag{3}$$

Here, F is a known function of x, m. and t, but the function $x(t)$ is unknown. The problem is to find a path $x = x(t), t0 \le t \le t1$ which optimize the functional J.

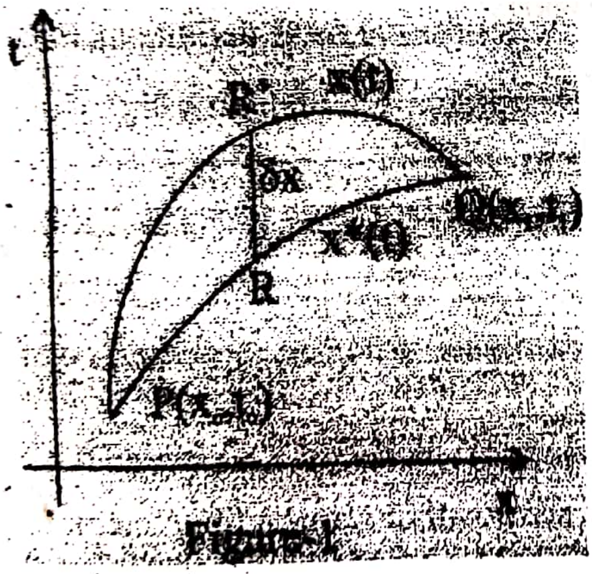

Figure-1

The value of J is different along different paths connecting the points $P(x0, t0)$ and $Q(x1, t1)$. We have to choose the path of integration $x(t)$ such that J has stationary value. We consider two paths out of infinite number of possible paths. The difference between the values of x for thee two paths for a given value of t, is the variation of x, which is denoted by δx and may be described by introducing a new function $\eta(t)$ and ε to describe the arbitrary deformation of the path and the magnitude of variation respectively.

The function $\eta(t)$ must satisfy the following two conditions :

(i) All varied paths must pass through the fixed points P & Q, i.e. $\eta(to) = \eta(ti) = 0$

(ii) $\eta(t)$ must be differentiable.

Let PRQ be the optimum path, i.e., J is optimum along PRQ and let it be given by $x = x^*(t)$. Also let a varied path is given by $x = x^*(t) + \delta x = x^*(t) + h(t).\varepsilon$.

Thus the value of J on $PR'Q$ is $J = \int_{t_0}^{t_1} F\left(x^*(t) + \eta(t)\varepsilon, \dot{x}^*(t)\varepsilon, t\right) dt$ where $\dot{\eta}(t) = \dfrac{d\eta}{dt}$

Hence, for a given $\eta(t)$, J is a function of ε only.

Therefore, $J(E) = \int_{t_0}^{t_1} F\left(x^*(t) + \eta(t)e, x^*(t) + \eta\cdot(t)\varepsilon, t\right) dt.$

The condition for extremum of $J(\varepsilon)$ is $\dfrac{dJ(\varepsilon)}{d\varepsilon} = 0$

Also from the figure, it is clear that $J(e)$ will be optimum where e=o.

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Thus the necessary condition for optimization of J is $\dfrac{dJ(\varepsilon)}{d\varepsilon} = 0$ for ε= o.

$$\text{Now } \frac{dJ}{d\varepsilon} = \int_{t_0}^{t_1} \left(\frac{\delta F}{\delta x} \frac{dx}{d\varepsilon} + \frac{\delta F}{\delta x} \frac{dx}{d\varepsilon} + \frac{\delta F}{\delta t} \frac{dx}{d\varepsilon} \right) dt$$

$$= \int_{t_0}^{t_1} \left(\frac{\delta F}{\delta x} \eta(t) + \frac{\delta F}{\delta x} \eta(t) + \frac{\delta F}{\delta t} \cdot 0 \right) dt$$

$$= \int_{t_0}^{t_1} \left\{ F_x\left(x^*(t) + \eta(t)\varepsilon, \dot{x}^*(t) + \dot{\eta}(t)\varepsilon, t\right) \cdot \eta(t) \right.$$

$$\left. + F_{\dot{x}}\left(x^*(t) + \eta(t)\varepsilon, \dot{x}^*(t) + \dot{\eta}(t)\varepsilon, t\right) \cdot \dot{\eta}(t) \right.$$

Since $\dfrac{dJ}{d\varepsilon} = 0$ for $e = 0$, therefore,

$$\int_{t_0}^{t_1}\left\{F_x\left(x^{*}t\right),x^{*},t)\eta\left(t\right)+F\dot{x}\left(x^{*}\left(t\right)t\right)\dot{\eta}\left(t\right)\right\}dt=0 \quad (4)$$

Now we consider the term $\int_{t_0}^{t_1} F_{\dot{x}}(x^{*}(t),\dot{x}^{*}(t))\dot{\eta}(t)dt$

$$=\left[F\dot{x}\int\eta\left(t\right)dt\right]_{t_0}^{t_1}-\int_{t_0}^{t_1}\left[\left\{\frac{d}{dt}(F\dot{x})\int\dot{\eta}\left(t\right)dt\right]dt\right.$$

$$=-\int_{t_0}^{t_1}\left(F_{\dot{x}}\right)\eta\left(t\right)dt \quad \left[\because \eta\left(t_0\right)=\eta\left(t_i\right)=0\right]$$

= Thus (4) becomes $=-\int_{t_0}^{t_1}\left(F_x\right)\eta\left(t\right)dt-\frac{d}{dt}F_{\dot{x}}\,\eta\left(t\right)dt=0$

or, $\int_{t_0}^{t_1}\left\{F_x-\frac{d}{dt}\left(F_{\dot{x}}\right)\right\}\eta\left(t\right)dt=0$

Since $\eta\left(t\right)$ is arbitrary deformation of the path; therefore

$$F_x-\frac{d}{dt}\left(F_{\dot{x}}\right)=0$$

or, $$\frac{d}{dt}\left(\frac{\delta F}{\delta\dot{x}}\right)-\frac{\delta F}{\delta x}=0 \quad (5)$$

This equation is known as Euler-Lagranges equation or simply Euler's equation and is used frequently in he study of calculus of variations.

Note-1 : When x is given at the end points to and t1, equ. (5) gives the necessary condition for the optimal path.

Note-2 : When x is not given at both the end points to and t_1, then η is completely arbitrary and its value at the end points are also arbitrary. In this case, the necessary Conditions for extremum of J are given by equation (5) together with $\left[\eta\frac{\delta F}{\delta\dot{x}}\right]_{t_0}^{t_1}=0, i.e., \frac{\delta F}{\delta\dot{x}}=0$ at $t=t_0$ & $t=t=t_1$.

Note-3 : When x is given at one end point only, say at $t = t_0$, then $\left[\eta \frac{\delta F}{\delta \dot{x}}\right]_{t_0}^{t_1} = 0$, gives $\eta(t_0) = 0$ and

$\frac{\delta F}{\delta \dot{x}} = 0$ at $t=t_1$. These end point conditions are called transversality condition.

Example 1 : Find he curve $x = x(t)$ which minimize the functional $J = \int_0^1 (\dot{x}^2 + 1)\,dt$ where, $x(0)=1$ and $x(1)=2$.

Solution : The Enler's equation for this problem, is given by

$$\frac{d}{dt}\left(\frac{\delta F}{\delta \dot{x}}\right) = 0 \text{ where } F(x, \dot{x}, t) = \dot{x}^2 + 1$$

Now $\left(\frac{\delta F}{\delta \dot{x}}\right) = 0$ and $\frac{\delta F}{\delta \dot{x}} = 2x.,$ so that $\frac{\delta F}{\delta \dot{x}}(2\dot{x}) - 0 = 0, or\ \ddot{x} = 0,$

or, $x = At + B$, Where A, B, are constants.

From the end conditions, we have $x(0) = 1 \Rightarrow A \cdot 0 + B = 1 \Rightarrow B = 1$

and $x(1) = 2 \Rightarrow A + B = 2 \Rightarrow A = 1$

Hence, the optimal path is $x(t) = t + 1$

and the corresponding value of J is $J \int_0^1 (x^2 + 1)\,dt = \int_0^1 (1 + 1)\,dt = 2$

Example 3 : Find the function $x(t)$ which minimizes the functional $J \int_0^1 (\dot{x}^2 + 1)\,dt$ where $x(0)=1$, but x can take any value at $t=1$

Solutin. Here also $F(x, \dot{x}, t) = \dot{x}^2 + 1$

Since x is not prescribed at the end point $t = 1$, we have the transversality conditoin $\frac{\delta F}{\delta \dot{x}} = 0$ $t = 1$, i.e. $\dot{x}(1) = 0$

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Now the euler's equation for this problem, gives $\frac{d}{dt}(2\dot{x})-0=0,$

or $\ddot{x}=0$, or, $x=At+B$, Where A, B are constants.

Using the conditoins $x(0)=1$ and $x.(1)=0$, we have $B=1$ and $A=0$

$\therefore x(t)=1$ for all t.

The optimum value of J along $x(t)=1$ is $J=\int_0^1(0+1)dt=1.$

3.1 Cost functional involving several dependent variables :

Let the integrand F be a function of one independent variable t and several dependent variables $x_1(t), x_2(t), ..., x_n(t)$. These dependent variables are functions of t only.

Now the problem is to find the functions $x_1(t), x_2(t), ..., x_n(t)$ such that the integral

$J=\int_{t_0}^{t_1}F(x_1,x_2,...,x_n,\dot{x}_1,...,\dot{x}_n;t)dt$ (6) may be statonary.

Let us consider two paths out of infinite number of paths between two points P and Q, Such that the difference between then may be described by the functions $\eta_k(t)$ and ε.

The function $\eta_k(t)$ must satisfy the following two conditions :

(i) All the neighbouring paths must pass throught the fixed points P and Q, i.e. $\eta_k(t_0)=\eta_k(t_1)=0$

(ii) $\eta_k(t)$ must be differentiable.

Let P & Q (Figure 1) be the path along which J has stationary value and $PR'Q$ be a neighbouring path. If x^*k and $\dot{x}^*k$ are the values of x_k and $\dot{x}_k$ along the optimum path, then the values of $x_1^*, x_2^*, ..., x_n^*$ in terms of $\eta_k(t)$ and ε are $x_i^*=x_i+\delta x_i=x_i+\varepsilon\eta_i(t)$, $i=1, 2, ..., n$.

If the integral has stationary value along PRQ, then

$$J(e)=\int_{t_o}^{t_1}F(x_1+\varepsilon\eta_1, x_2+\varepsilon\eta_2,...,x_n+\varepsilon\eta_n,\dot{x}_2+\varepsilon\dot{\eta}_2...,x.n+\varepsilon\dot{\eta}_n;t_0dt \quad (7)$$

Now, $J(\varepsilon)$ is statonary for $\varepsilon=0$, as in case of single dependent variable. Differentiating (7) with respect to e, we have $\frac{dJ}{d\varepsilon} = \int_{t_0}^{t_1} \sum_k \left(\frac{\partial F}{\partial x^* k} \cdot \frac{dxk^*}{d\varepsilon} + \frac{\partial F}{\partial \dot{x}^* k} \cdot \frac{d\dot{x}^* k}{d\varepsilon} \right) dt$

$$= \int_{t_0}^{t_1} \sum_k \left(\frac{\partial F}{\partial x^* k} \eta_k(t) + \frac{\partial F}{\partial \dot{x}^* k} \dot{\eta}_k(t) \right) dt$$

Now, $\left[\frac{dJ(\varepsilon)}{d\varepsilon} \right]_{\varepsilon=0} = \int_{t_0}^{t_1} \sum_k \left(\eta_k \frac{\partial F}{\partial x_k} + \dot{\eta}_k \frac{\partial F}{\partial \dot{x}_k} \right) dt$

Since $(x_k^*)_{t=0} = x_k$ and $(\dot{x}_k^*)_{\varepsilon=0} = \dot{x}_k$.

The condition for the path along which J has stationary value is $\left[\frac{dJ(\varepsilon)}{d\varepsilon} \right]_{\varepsilon=0} = 0$

Hence $= \int_{t_0}^{t_1} \sum_k \left(\eta_k \frac{\partial F}{\partial \dot{x}_k} + \dot{\eta}_k \frac{\partial F}{\partial \dot{x}_k} \right) dt = 0$ (8)

Integrating the second term by parts,

$$= \int_{t_0}^{t_1} \dot{\eta}_k \frac{\partial F}{\partial \dot{x}_k} dt = \left[\dot{\eta}_k \frac{\partial F}{\partial \dot{x}_k} \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \eta_k \frac{d}{dt} \left[\frac{\partial F}{\partial \dot{x}_k} \right] dt - \int_{t_0}^{t_1} \eta_k \frac{d}{dt} \left[\frac{\partial F}{\partial \dot{x}_k} \right] dt$$

[since $\eta_k(t_0) = \eta_k(t_1) = 0$]

Using this result, equation (8) becomes

$$\int_{t_0}^{t_1} \sum_k \left[\eta_k \frac{\partial F}{\partial x_k} - \eta_k \frac{d}{dt} \frac{\partial F}{\partial \dot{x}_k} \right] dt = 0.$$

or, $\int_{t_0}^{t_1} \sum_k \eta_k \left[\frac{\partial F}{\partial x_k} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}_k} \right] dt = 0.$

Since hm are perfectly arbitrary and independent of one another, the terms within the square bracket of equation (9) are separately zero. Thus,

$$\frac{\partial F}{\partial x_k}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_k}\right)=0, k=1,2,3,...,n. \tag{10}$$

Equation (10) represents a whole set of Eüler-Laangrange equations each of which must be satisfied for an extreme value.

Example 3 : Find x and y as functions of t, so that

$$J=\int_{t_0}^{t_1}\left[\frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right)-mgy\right]dt$$

May have stationary value. It may be assumed that x and y are given at to and $t1$.

Solution : Here, $F=\frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right)-mgy$ contains two dependent variables x and y

Thus, Enler-Lagrange equations are

$$\frac{\partial F}{\partial x}-\frac{d}{dt}\left(\frac{dF}{d\dot{x}}\right)=0 \text{ and } \frac{\partial F}{\partial y}-\frac{d}{dt}\left(\frac{dF}{d\dot{y}}\right)=0$$

These equations reduce to

$$\frac{d}{dt}(\dot{x})=0 \text{ and } \frac{d}{dt}(m\dot{y})+mg=0,$$

Which give $x=c_1t+c_2$ and $y=-\frac{1}{2}gt^2+d_1t+d_2$,

Let $x(t_0)=x_0, d_2$ and the required solūtion is $x=c_1t+t_2$ and $y=-\frac{1}{2}gt^2+d_1t+d_2$,

3.2 Optimization with constraints :

Let the problem is to find the path $x_i=x_i(t), to \le +t_1, (i=1,2,...,n)$ which maximizes or minimizes the cost functional

(11)

$$J = \int_{t_0}^{t_1} F\left(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n, t\right) dt$$

(12)

Where xi's satisfy the constraints.

$$g\left(x_1, x_2, \ldots, x_n; \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n; t\right) = 0 \; \& \; h\left(x_1, x_2, \ldots, x_n; \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n; t\right) = 0$$

We solve this problem in the same way as we solve on ordinary extremum of a function by introducing two Lagrangian λ and μ and forming the angmented cost functional as

$$J^{*} = \int_{t_0}^{t_1} \left(F + \lambda g + \mu h\right) dt$$

Now, we find the extremum of these functional as before and the lagranges multipliers are found from the Euler's equationsn for this new functional and the given constraints.

Example 4 : Find the stationary path $x = x(t)$ for the functional $J = \int_{t_0}^{t_1} \left[1 + \left(\frac{d^2x}{dt^2}\right)\right] dt$

Subject to the boundary conditions

$x(0) = 0$

$\dot{x}(0) = 1$

$x(1) = 1$

$\dot{x}(1) = 1$

Solution. To eliminate the second order derivative form the functional J, we introduce two variables, namely $x_1 = x$ and $x_2 = x$

as it is supposed to be a function of t, x and $\dot{x}$ only. Thus the problem reduces to funding the extremum of the functional $J = \frac{dF}{d\dot{x}}$

with the boundary conditions $x_1(0) = 0$,

$x_2(0) = 1$

$x_1(1) = 1$

$x_1(1) = 1$

$m_2(1) = 1$

and the constraint $x_2 - \dot{x}_1 = 0$

The constraint is incorporated into the functional J, using Lagranges multiplier λ and we formulate the augmented functional as

$$J^* = \int_0^1 \left[1+\dot{x}_2^2+1\left(x_2-\dot{x}_1\right)\right]dt$$

To find the optimal path of this functional, we solve the corresponding Euler's equations given below.

For x_1 : $\frac{\delta F}{\delta x_1}-\frac{d}{dt}\left(\frac{\delta F}{\delta \dot{x}_1}\right)=0$ where $F=1+\dot{x}_2^2+\lambda\left(x_2-\dot{x}_1\right)$

or, $\frac{d}{dt}(-1)=0$

or, $\dot{\lambda}=0$

or, λ = constant indendent of t

For x_2 : $\frac{\delta F}{\delta x_2}-\frac{d}{dt}\left(\frac{\delta F}{\delta \dot{x}_2}\right)=0$

or, $\lambda-2\ddot{x}_2=0$

or, $\ddot{x}_2=\frac{\lambda}{2}=A(say)$

or, $\dot{x}_2=At+B$

or, $x_2=\frac{At^2}{2}+Bt+C$

$\therefore \dot{x}_1=\frac{At^2}{2}+Bt+C$

or, $x_1=\frac{At^3}{6}+\frac{Bt^3}{2}+Ct+D$

Where A, B, C, D are constants.

Now, using the boundary condition $x1(0)=0$, we have $D=0$ and $x_2(0)=1$ gives $C=1$.

$$\therefore x_2(t)=\frac{At^2}{2}+Bt+C$$

$$\text{or, } x_1=\frac{At^3}{6}+\frac{Bt^2}{2}+t$$

Again $x_1(1)=1$ gives $\frac{A}{6}+\frac{B}{2}=0$

and $x_2(1)=1$ gives $\frac{A}{2}+B=0$, so that $A=B=0$

and thus $x(t)=x_1(t)=t$

Hence, the optimal path is given by $x(t)=t, 0\le t\le 1$

and he corresponding value of the functional is $J^*=\int_0^1\left[1+\left(\frac{d^2x}{dt^2}\right)^2\right]dt=1$

4. Optimal Control.

With the help of the following examples, the problems of optimal control are illustrated in this section.

Example 5 : A particle is attached to the lower end of a vertical spring whose other end is fixed, is oscillating about its equillibrium position. If x denote the particule's displacement from the equilibrium position, the governing differential equation for this motion is $\ddot{x}=-w^2x$.

If the particle is at its maximum displacement $x=a$ at time $t=0$ and at this instant of time, a force is per unit mass is applied to the particle in order to bring the particle to rest when its displacement is zero, find such a force u.

Solution. After the application of the force u at $x=a$ and $t=0$, the governing differential equation becomes $\ddot{x}=-w^2x+u$.

The system can be represented in the form of first order differential equations by using the variables $x_1=x$ and $x_2=\dot{x}_1=\dot{x}$

as $\dot{x}_2=-w_2x_1+u$ and $x_2=\dot{x}_1$

With the conditions $x_1(0) = a$ and $x_2(0) = 0$

Now the control problem is to choose u so that the system (x_1, x_2) moves from $(a, 0)$ at $t = 0$ to $(0, 0)$ at some subsequent time.

As an initial guess, let us assume u = constant = c.

Then we have $\dot{x}_2 = -W^2 x_1 + C$

or $\ddot{x}_1 = -w^2 x_1 + C$

or, $x_1 = A\cos wt + B\sin wt + \frac{C}{w^2}$

and $x_2 = \dot{x}_1 = -Aw\sin wt + Bw\cos wt.$

At $t = 0$, we have $x_1 = a$ and $x_2 = 0$

$$\therefore B = 0 \text{ and } A = a - \frac{C}{w^2}$$

$$\therefore x_1 = \left(a - \frac{C}{w^2}\right)\cos wt + \frac{C}{w^2}$$

$$\text{and } x_2 = \left(a - \frac{C}{w^2}\right)$$

The value of x_1, i.e. velocity will again becomes zero when $t = \frac{\pi}{w}$. At that time

$x_1 = -\left(a - \frac{C}{w^2}\right) + \frac{C}{w^2} = -a + \frac{C}{w^2}$. By the given condition, x_1, i.e., displacement must be zero.

$$\therefore -a + \frac{2C}{w^2} = 0$$

$$\text{or, } C = \frac{aw^2}{2}$$

Hence the control variable, i.e. force $u = \frac{aw^2}{2}$ takes the system from $(a, 0)$ at $t = 0$ to $(0, 0)$ at $t = \frac{\pi}{w}$ and the system is controllable.

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Example 6 : An electrochemical system is modelled by the differential equation $\ddot{x} = -\dot{x} + u$, where x and u are functions of time t.

Minimize the cost functional $J\frac{1}{2}\int_0^{\alpha}\left(x^2 + \alpha\dot{u}^2\right)dt$ when α is a disposable constant by chosing the control variable properly.

Solution : As before, we convert the given second order differential equation to the first order differential equation by introducing the state variables x_1 and x_2

as $x_1 = x$

and $x_2 = \dot{x}_1 = \dot{x}$

Hence, the variabels x_1 and x_2 satisfy the equations

$\dot{x}_2 = -x_2 + u$

and $\dot{x}_1 = x_2$

The end conditions for an electrochemical system are that x and x are given at $t = 0$ and both tend to zero as $t \to a$

In terms of state variables, the above end conditons can be represented as

$x_1(0) = a$ (say)

$x_2(0) = b$ (say)

$x_1 \to 0$ and $x_2 \to 0$ as $t \to a$

Where, a and b are constants.

Hence the problem reduces to minimizing

$$J = \frac{1}{2}\int_0^{\alpha}\left(x_1^2 + \alpha u^2\right)dt$$

subject to the constraints $\dot{x}_2 = -x_1 + u$ and $\dot{x}_1 = x_2$ along with the above end conditions.

We form the angmented cost functons with the help of the lagranges multipliers λ and μ as.

$$J^* = \frac{1}{2}\int_0^{\alpha}\left(x^2 + \alpha u^2\right)dt$$

The Euler's equation for the state variables x1 and x2 and the control variable u for the above functional are:

for $x_1 : \frac{\partial F}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_1}\right)$

or, $x_1 - \frac{d\lambda}{dt} = 0$

or, $x_1 = \lambda$

for $x_2 : \frac{\partial F}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_1}\right) = 0$

or, $-\mu + \lambda - \frac{d\lambda}{dt} = 0$

or, $\mu = \lambda - 1$.

for $u; \frac{\partial F}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_1}\right) = 0$

or, $\alpha u - \lambda = 0$ or $\lambda = \alpha u$.

Combining these three relations $x_1 = \lambda, \varpi = \lambda - \dot{\lambda}$ and $\lambda = \alpha u$ along with the constraints, we can eliminate λ and μ and and solve for x_1, x_2 and u as follows.

$$\frac{d^4 x_1}{dt^4} = \frac{d^3}{dt^3}\left(\frac{dx_1}{dt}\right) = \frac{d_3 x_1}{dt_3} = \frac{d^2}{dt^2}\left(\frac{dx_1}{dt}\right) = \frac{d^2}{dt^2}(-x_2 + u)$$

$$= -\frac{d}{dt}\left(\frac{dx_1}{dt}\right)u$$

$$= \frac{d}{dt}(-x_2 + u) + u$$

$$= \dot{x}_2 - \dot{u} + \ddot{u}$$

$$= \ddot{x}_1 \frac{\lambda}{\alpha} + \frac{\lambda}{\alpha} (\because \lambda = \alpha u)$$

$$= x_1 - \frac{1}{\alpha}\frac{d}{dt}(\lambda - \dot{\lambda})$$

$$= \ddot{x}_1 - \frac{1}{\alpha}\mu.(\because \mu = \lambda - \dot{\lambda})$$

$$= \ddot{x}_1 - \frac{1}{\alpha}x_1 (\because \dot{x}_1 = \dot{\mu})$$

$$\therefore \frac{d^4x_1}{dt^4} - \frac{d^2x_1}{dt^2} + \frac{1}{\alpha}x_1 = 0$$

or, $\left(D^4 - D^2 + \frac{1}{\alpha}\right)x1 = 0$, Where $d \equiv \frac{d}{dt}$

This indicates that the solution of $x1, x2$ and the control variable depends on the choice of the disposable constant α. For simplicity, let us choose $\alpha = \frac{1}{4}$. Then the above equatin reduces to

$$\left(D^4 - D^2 + \frac{1}{4}\right)x_1 = 0$$

or, $\left\{\left(D + \frac{1}{\sqrt{2}}\right)\left(D - \frac{1}{\sqrt{2}}\right)\right\}^2 x_1 = 0$

or, $x_1(t) = (c_1t + c_2)e^{\frac{1}{\sqrt{2}}t} + (c_3t + c_4)e^{\frac{1}{\sqrt{2t}}}$

Where c_1, c_2, c_2, c_4 are constants which are to be evaluated with the help of the end conditions.

As $x_1 \to 0$ as $t \to \alpha$, We have $c_3 - c_4 = 0$

$\therefore x_1(t) = (c_1t + c_2)e^{\frac{1}{\sqrt{2}}t}$, So that $x_2 = x_1 = c_1e^{\frac{1}{\sqrt{2t}}} - \frac{1}{\sqrt{2}}(c_3t + c_4)e^{\frac{1}{\sqrt{2t}}}$

Now, $x_1(0)=a$ gives $c_2 = a$, and $x_0(0) = b$ gives $c_1 - \frac{1}{\sqrt{2}}c_2 = b$

$$\therefore c_1 = b + \frac{1}{\sqrt{2}} \text{ since } c_2 = a$$

$$\therefore x_1(t) = \left\{\left(b + \frac{a}{\sqrt{2}}\right)t + a\right\} e^{-\frac{1}{\sqrt{2}}t}$$

$$\text{and } x_2(t) = \left(b + \frac{a}{\sqrt{2}}\right) e^{-\frac{1}{\sqrt{2}}t} - \frac{1}{\sqrt{2}}\left\{\left(b + \frac{a}{\sqrt{2}}\right)t + a\right\} e^{-\frac{1}{\sqrt{2}}t}$$

$$= \left\{b - \left(b + \frac{a}{\sqrt{2}}\right)\frac{t}{\sqrt{2}}\right\} e^{-\frac{1}{\sqrt{2}}t}$$

$$\therefore u = \dot{x}_2 + x_2 = \left[\frac{1}{2}\left(1-\sqrt{2}\right)\left(b + \frac{a}{\sqrt{2}}\right)t - \left(\sqrt{2}-1\right)b - \frac{a}{2}\right] e^{-\frac{1}{\sqrt{2}}t}$$

Thus, we have determined the control variable u which minimizes the functional J.

Representation of the solution by block diagram :

The solution of the electrochemical rpocess control problem can be illustrated by a block diagram nothing that u is of the form $u = a_1 x_1 + a_2 x_2$,

Where a_1, a_2 are constants to be determined.

Comparing the co-effcients of t and constant terms from both sides, we have

$$\frac{1}{2}\left(1-\sqrt{2}\right)\left(b + \frac{a}{\sqrt{2}}\right) = \left(b + \frac{a}{\sqrt{2}}\right)a_1 - \frac{1}{\sqrt{2}}\left(b + \frac{a}{\sqrt{2}}\right)a_2$$

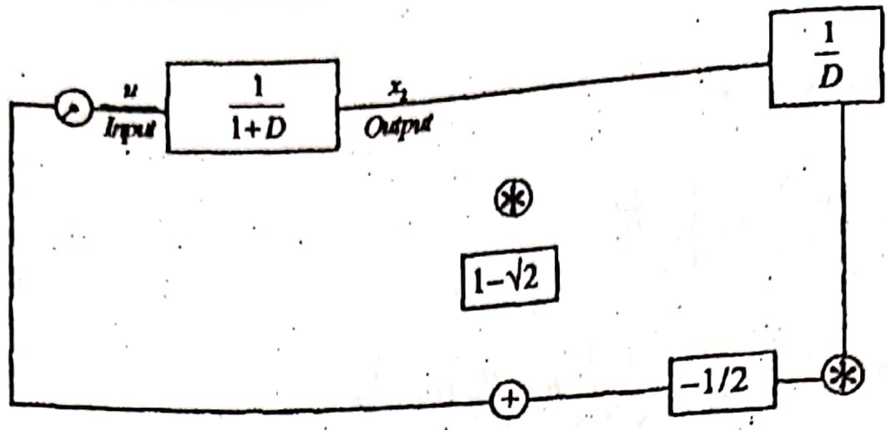
$$\text{i.e., } 2a_1 = \sqrt{2}a_2 = \left(1-\sqrt{2}\right)$$

$$\text{and } -\left(\sqrt{2}-1\right)b - \frac{a}{2} = aa_1 + ba_2$$

$$\text{i.e., } aa_1 + ba_2 = -\frac{a}{2} + \left(1-\sqrt{2}\right)b$$

Slowing, we get $a_1 = \frac{1}{2}$ and $a_2 = 1-\sqrt{2}$

$$\therefore u(t) = -\frac{1}{2}x_1(t) + \left(1-\sqrt{2}\right)x_2(t)$$



Block diagram

4.1 Bang Bang control

A concept of Bang Bang control is illustrated by the following simple physical example.

Let a car is drivced from a stationary position on a horizontal way to a stationary position in a garage moving a total distance 'a'. The available control for the driver are the accelarator and the break (for simplicity we consider no grear change). The corresponding equation of motion for the car is

$$\frac{d^2x}{dt^2} = f \tag{13}$$

Where, $f = f(t)$ represents the acceleration or decelertion clearly, f will be subjected to both lower and upper bounds, i.e/, maximum acceleration and maximum deceleration so that $-\alpha \le f(t) \le \beta$ (14)

Where β is the maximum possible acceleration and a is the maximum possible deceleration. Now the problem is to solve the equation (13) subject to the constraint (14) with the initial conditions.

$$\left.\begin{aligned} x(0) = 0, \left(\frac{dx}{dt}\right)_{t=0} = 0 \\ \text{and } x(T) = a, \left(\frac{dx}{dt}\right)_{t=T} = 0 \end{aligned}\right\} \tag{15}$$

Where T is the time of travel. Now the problem is to find out the control f which accomplishes the operation in a minimum time.

The time of travel T can be expressed as

$$T = \int_0^T dt = \int_0^a \frac{dt}{dx}dx = \int_0^a \frac{1}{\frac{dx}{dy}}dx = \int_0^a \frac{1}{v}dx$$

Where v is the velocity of the car. Regarding v as a function of x, we can have the end conditions in terms of v as $v(0)$ and $v(a)=0$

$$\text{Now, } f = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}(v) = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}\cdot v$$

$$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$= \frac{dg}{dx}$$

Where, $g = \frac{1}{2}v^2$ or, $v = \sqrt{2g}$

$$\therefore T = \int_0^a \frac{1}{v}dx = \int_0^a \frac{1}{\sqrt{2g}}dx \qquad (16)$$

With $g(0)=0$ and $g(a)=0$

$$\text{Also we have } f = \frac{dg}{dx}, \text{ or } \frac{dg}{dx} - f = 0 \qquad (17)$$

The reduced problem is to find the control f which minimizes the functional T given by (16) subject t the constraint (17) and the inequality constraint (14) along with the end conditions $g(0)=0$ and $g(0)=0$.

Now we change the inequality constraint (14) into the equality constraint by introducing another control variable z where, $z_2 = (f+\alpha)(-f+\beta)$ (18)

z being a real variable and if the inequality constraint (14) is satisfied then $z^2 \geq 0$.

Hence the problem is to minimize T given by (16) subject to the equality constraints (17) and (18) along with the end condition $g(0)=0$ and $g(a)=0$.

We formulate the angmented cost functional as

$$T^{*} = \int_0^a \left[\frac{1}{\sqrt{2g}} + \alpha\left(\frac{dg}{dx} - f\right) + \mu\{z^2 - (f+\alpha)(-f+\beta)\} \right] dx$$

Where g is the stae variable, f and z are control variable and λ, μ are Lagranges multipliers. The optimal path will satisfy the following Euler's equalitions for g:

$$\frac{\partial F}{\partial g} - \frac{d}{dx}\left(\frac{\partial F}{\partial g}\right) = 0 \text{ Where } F = \frac{1}{\sqrt{2g}} + \lambda(g^1 - f) + \mu\{z^2 - (f+\alpha)(-f+\beta)\} \quad (20)$$

or, $-(2g)^{-3/2} - \frac{d\lambda}{dx} = 0$

$$\text{for } f: \frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z^1}\right) = 0 \quad (21)$$

or, $-\lambda + 2\mu f + \mu\alpha - \mu\beta = 0$

$$\text{for } f: \frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z^1}\right) = 0 \quad (22)$$

or, $2\mu z = 0$

From (22), We have either $z = 0$ or $\mu = 0$

If $m = 0$ from (21) $\lambda = 0$ and from (20). $g \to \alpha$ which is clearly not possible, T will be zero if $g \to a$.

Hence, $m \neq 0$, so, that $z = 0$

$\therefore (f+\alpha)(-f+\beta) = 0$

Which gives $f = -\alpha$ or $f = \beta$.

Thus on the optimal path, the acceleration and the deceleration forces take their maximum values.

Hence from the nature of the problem, we may conclude that initially maximum accleration β is applied and then after time $t = 7$ (say), the maximum deceleration $-\alpha$ is applied to bring the car to rest (velocity is zero) at time $t = T$.

$\therefore$ The equation of motions are

$$\frac{d^2x}{dt^2} = f = \begin{cases} \beta, 0 \le t \le \tau \\ -\alpha, \tau \le t \le T \end{cases} \tag{23}$$

We assume that by a switching device the change from β to $-\alpha$ takes place at a time $t = 7$ which is to be determined.

From (23) we get, on integration.

$$\frac{dx}{dt} = \begin{cases} \beta t, 0 \le t \le \tau \\ -\alpha(t-T), \tau \le t \le T \end{cases}$$

Again integrating,

$$x(t) = \begin{cases} \frac{1}{2}\beta t^2 \text{for } 0 \le t \le \tau \\ -\frac{\alpha}{2}(t-T) + a \text{ for } \tau \le t \le T \end{cases}$$

Now, $d(t)$ and $\frac{dx}{dt}$ must be continuous at $t = t$ then we have

$$\beta t = a(t-T)$$

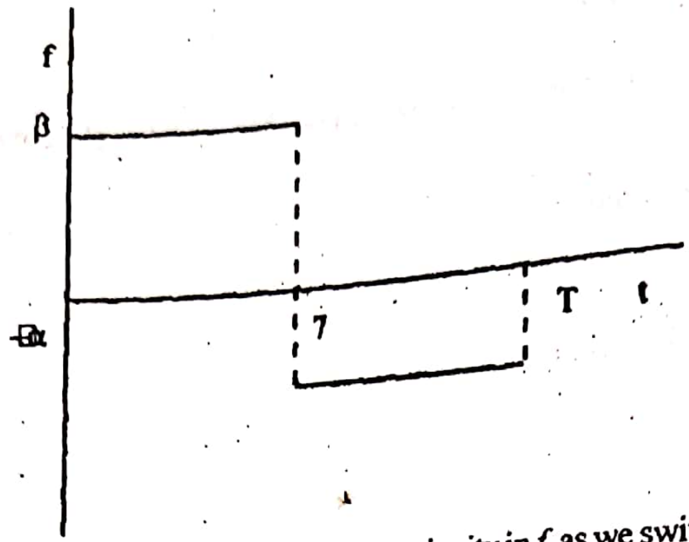
and $\frac{\beta}{2}\tau^2 = \frac{\alpha}{2}(\tau - T)^2 + a$

Slowing, we get $\tau = \sqrt{\frac{2aa}{\beta(\alpha+\beta)}}$ (24)

and $T = \sqrt{\frac{2a(\alpha+\beta)}{\alpha\beta}}$ (25)

Hence the minimum time to bring the car in the stationary position at a distance 'a' is given by (25) and the optimal control to be applied on the car is given by equation (23) where t is given by equation (24).

Graphically control can be represented as follows.



The graph reverals the fact that at $t = 7$, there is a discontinuity in f, as we switch it over from its maximum positive value to its minimum negative value. For this reason, this type of control is referred to as barg bang control.

Example 7 : Angular motion of a ship is described by an equation $\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt} = p$

Where p is the rader setting which is subject to the constraint $|p| \le 1$.

To change the angle $\theta = \alpha$ to $\theta = 0$, it is required to make $\frac{d\theta}{dt} = 0$. Find p to minimize the time taken for correction.

Solution : Let the time taken to change the required direction is T, which is given by

$$T = \int_{t_0}^{T} dt = \int_{\theta=\alpha}^{0} \frac{dt}{d\theta} = \int_{\alpha}^{0} \frac{1}{\frac{d\theta}{dt}} d\theta = \int_{t_0}^{t_1} \frac{1}{w} d\theta$$

Where $w = \frac{d\theta}{dt}$. Now regarding w to be a function of q, we have a state variable w which is the independent variable. Expressing p in terms of w, we have

$$p = \frac{d^2\theta}{dt^2} + \frac{d\theta}{dt}$$

$$= \frac{d}{dt}\left(\frac{d\theta}{dt}\right) + \frac{d\theta}{dt}$$

$$= \frac{dw}{dt} + w$$

$$= \frac{dw}{d\theta} + \frac{d\theta}{dt} + w$$

$$= w\left(\frac{dw}{d\theta} + 1\right)$$

Where p is the control variable.

Now the problem is to choose w to minimize T subject to the constraints $p = w\left(\frac{dw}{d\theta} + 1\right)$ and $|p| \le 1$.

The inequality constraint $|p| \le 1$ can be replaced by introducing a new control variable z as $x^2 = (p+1)(-p+1)$

Now We construct the augented cost functional as

$$T^* = \int_{a}^{0}\left[\frac{1}{w} + \lambda\left\{p - w\left(1 + \frac{dw}{d\theta}\right)\right\} + \mu\left\{z^2 - (p+1)(-p+1)\right\}\right]d\theta$$

Where λ and μ are Lagranges multipliers.

For the optimal path, the Euler's equations are given below.

For W : $\frac{dF}{dw} - \frac{d}{d\theta}\left(\frac{dw}{d\theta = 0}\right)$

Where, $F = \frac{1}{w} + \lambda\{p - w(1+w)\} + \mu\{z^2 - (p+1)(-p+1)\}$

or, $-\frac{1}{w^2} - \lambda(1+w') - (-\lambda w') + \lambda' w = 0$

or, $\lambda = \frac{1}{w^2} + \lambda' w$

or, $\lambda' w - \lambda = \frac{1}{w^2}$

For : $P : \frac{\partial F}{\partial p} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial p'}\right) = 0$

or, $\lambda + 2\mu p = 0$

or, $\lambda = -2\mu p$

For $z : \frac{\partial F}{\partial p} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial p'}\right) = 0$

or, $2\mu z = 0$

As in the previous problem, $\mu = 0$ is not possible, so that $z = 0$, i.e. $(p+1)(-p+1) = 0$, or, $p = +1$.

Now q at $t = 0$, we have $\theta = \alpha$ and $\frac{d\theta}{dt} = 0$

Therefore, initially the control should be -1 and at the and of the path it should be $p = 1$. Assuming that there is only one switch which changes the control from $p = -1$ to $p = 1$, we determine the time when this swtich over takes place. For $p = -1$, the given differential equation can be written as

$$-1 = w\left(1 + \frac{dw}{d\theta}\right)$$

or, $w\frac{dw}{d\theta} = 1 + w$

or, $\frac{w}{1+w}\frac{dw}{d\theta} = 1$

Integrating we get.

$w - \log(1+w) = -\theta + c_1, c_1$ is constant.

At $\theta = d, w = 0$, Therefore, $c_1 = a$ and so, $\theta = -w + \log(1+w) + a$

Similarly, for $p = 1$, we have

$$1 = w\left(1 + \frac{dw}{d\theta}\right)$$

or, $\frac{w}{1-w}dw = d\theta$

Integrating we get,

$-w-\log(1-w) = q + c_2, c_2$ is a constant.

At $\theta = 0$, Therefore $c_2 = 0$ and so, $\theta = -w - \log(1-w)$.

The switch over for $p = -1$ to $p = 1$ occurs when the value of q from the above expressions are equal

$$\therefore e., -w + \log(1+w) + a = -w - \log(1-w)$$

or, $\log(1-w^2) + a = 0$

or, $1 - w^2 = e - a$

or, $w = \sqrt{1-e^{-\alpha}}$

Substituting this value of w in

$\theta = -w - \log(1-w)$, we have the corresponding value of θ as $-\sqrt{1-e^{-\alpha}} - \log\sqrt{1-e^{-\alpha}}$

Thus the control p is given by

$$p = \begin{cases} -1 \text{ for } \theta c \le \theta \le \alpha \\ 1 \text{ for } 0 \le \theta \le \theta c \end{cases}$$

Where $\theta c = -\sqrt{1-e^{-\alpha}} - \log\left(1-\sqrt{1-e^{-\alpha}}\right)$

5. Pontryagin's Maximum Principle.

In this section, we present the theoretically interesting and practically useful theorem due to pontryagin in connection with the optimal contrd theory.

Let the given control system is governed by the ordinary differential equation.

$$x\cdot(t) = f(x(t)), u(t), t \ge 0$$

Where $x(t) = (x_1, x_2, ..., xn)(eR^w$ and $u(t) = (u_1, u_2, ..., um)erm, x(t)$ being the state variable vector and $u\,(t)$ being the control variable vector having the range set $A \le R^m$ and A be the set of all possible controls.

Let the initial condition be given by

$$x(0) = x^0 = (x_1^0, x_2^0, ..., x_n^0).$$

and the pay of functional be

$$J\int_0^T r(x(t),u(t))dt + g(x(T))$$

Where the terminal time $T > 0$, running payoff $r : Rn \times A \rightarrow R$ and terminal payoff $G : Rn \rightarrow R$ are given.

Then the problem is to find a control v^* such that J in minimized over all $U e A$.

Let us define the ontrol theory Hamiltonian by the function $H(x, p, a) f(xma).p + r(x, a)$, where $x, p, e\ Rn$ and $ae A$. Here the newly introduced variable vector p is called the costate. It may be considered as a sort of Lagranges multiplier.

Now we assume that U^* be the optimal control for the problem under consideration and x^* be the corresponding troaectory. Then the pontryagin's Maximum principle asserts the existence of a function $p^* : [0, T] \rightarrow R^n$ such that

$$\dot{x}^* = \Delta p H(x^*(t), p^*(t), U^*(t),$$

$$\dot{p}^*(t) = -\Delta p H(x^*(t), p^*(t), U^*(t)),$$

and $H(x^*(t), p^*(t), U^*(t)) = \max H(x^*(t), p^*(t), a)$, which signifies the name of the theorem.

Also we have the terminal conditon .

$$p^*(T) = \Delta g(x^*(T).$$

Furthermore, $H(x^*(t), p^*(t), u^*(t))$ is independent of t.

We now illustrate the pontryagin's Maximum principle by the following model for optimal consumption in simple economy.

Let $x(t)$ = output of the economy at time t and $u(t)$ = fraction of output reinvested at time t. We have the constraints $0 \le u(t) \le 1, i.e., A = [0, 1]$. The economy evolves according to the dynamics

$$\left.\begin{array}{l} x(t) = kx(t)u(t), 0 \le t \le T \\ x(0) = x^0 \end{array}\right\}$$

Where k is a constant, known as the growth factor. In this case we set the growth factor $k = 1$. We want to maximize the total consumption

$$J = \frac{\delta F}{\delta \dot{x}}$$

The problem is to find an optimal control u^*. We apply pontryagin's Maximum principle, and to simplify notation we will not write the superscripts * for the optimal control, trajectory, etc. Here we have $n = m = 1$,

$f(x,a) = xa, g = 0, r(x,a) = (1-a)x$; and therefore

$H(x,p,a) = f(x,a).p + r(x,a) = xap + (1-a)x = x + ax(p-1).$

The dynamical system is

$$\dot{x}(t) = \frac{\delta H}{\delta p}(x,p,u) = ux$$

and $\dot{p}(t) = -\frac{\delta H}{\delta x}(x,p,u) = -1 - u(p-1).$

The second equation of the above system is called the adjoint equation.

the term inal conditions gives $p(T) = \frac{\delta g}{\delta x}(x(T)) = 0$

Lastly, the maximality principle asserts

$H(x, p, u) = \max H(x, p, a)$

ae [, 1]

$= \max \{x(t) + ax(t)(p(t) - 1)\}.$

ae [0, 1]

Since $x(t) > 0$, at each time t the control value $u(t)$ must be selected to maximize $a(p(t)-1)$ for $0 \leq a \leq 1$.

Thus $u(t) = \begin{cases} 1 \text{ uf } p(t) > 1 \\ 0 \text{ if } p(t) \leq 1 \end{cases}$

Hence if p is known, the optimal control can be designed at once. So next we must solve for the constate p.

We have from the adjoint equation and the terminal condition.

$p(t) = 1 - u(t)(p(t) - 1),) \leq t \leq T$ and $p-(T) = 0$

Since $p(T) = 0$, we deduce by continuity that $p(t)<1$ for t close to T, $t<T$. Thus $u(t)=0$ for such values of t.

Therefore $p(t) = -1$ and consequently $p(t)=T-t$ for times t in this interval. So we have that $p(t)= T-t$ so long as $p(t)<1$ and this holds for $T-1<t<T$.

But for times $t<T-1$, with t near $T-1$, we have $u(t)=1$ and so $p(t)=-1-(p(t)-1)$

Since $p(T-1)=1$, we see that $p(t) = e^{T-1-t} > 1$ for all times $0 \leq t \leq T-1$. In particular there are no switches in the control over this time interval.

Rstoring the superscript * to our notation, we deduce that the optimal control is

$$u^*(t) = \begin{cases} 1 \ if \ 0 \le t \le t^* \\ 0 \ if \ t^* \le t \le 1 \end{cases}$$

For the optimal switching time $t^* = T - 1$.

The solution of the production and consumption model shows that it is also an example of a Bang Bang control.

6. Conclusions

In this module, we have studied the theory of optimal control. Perforamnce indices for different types of control problems are discussed. We have used two techniques, namely -- the calculus of variations and the Pontryagins Masximum Principle to solve the contrll problems. Using the calculus of variations, the Eubr-Lagrange equations are deduced and those are solved to obtain the optimal trajectory. Pontryanins Maximum Principle asserts the existence of a function called the costate, which together with the optimal trajectory satisfy some equations which are analogus with the classical Hamiltoman dynamics. Bang Bang controls are illustrated through examples.

7. Exercise

(i) obtain the performance index for general optimal control systems. When a control problem is said to be of- a) Mayer type b) Bolza type c) Lagrange type?

(ii) Prove that $J = \int_{x_0}^{x_1} F(y, y', x)\,dx$ will be minimum only when $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$

(iii) Show that the functional $J = \int_{x_0}^{x_1} \frac{(1+y^2)}{y'^2}dx$ will be extremum of $y = \sinh(c_1x + c_2)$ where c_1, c_2 are arbitrary constants.

(iv) Find the least value of the integral $\int_A^B \frac{1}{y}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx$ where A is $(-1, 1)$ and B is $(1, 1)$.

(v) Obtain the necessary condition for the existence of a sationary value of the functional

$$J = \int_{x_0}^{x_1} F(y_1, y_2, \ldots, yn, y_1', y_2], \ldots, yn', x)dx)$$

(vi) Find x and y as functions of t, so that $\int_{t_0}^{t_1}\left[\frac{m}{2}\left(x^2+y^2\right)-mgy\right]dt$

May have stationary valuc. It may be assumed that x and y are given at to and t_1.

(vii) A system is described by $\dot{x}_1=-2x_1+u$ and the control $u(t)$ is to be chosen in order to minimize $\int_0 u_2 dt$.

Show that theoptimal control which transfer the system from $x_1(0)=1$ to $x_1(1)=1$ is given by $u^*=-\frac{4e^{2t}}{e^4-1}$

(viii) An electrochemical system is characterised by the ordinary differential equations $\frac{dx_1}{dt}=x_2$ and $\frac{dx_2}{dt}=x_2=u$

Where u is the control variable chosen in such a way that the cost functional $\frac{1}{2}\int_0^{\alpha}\left(x_1^2+4u^2\right)dt$ is minimized.

Show that, if the boundary conditions satisfied by the state variabels are $x_1=(0)=a, x_2(0)=b$ whether a, b are constants and $x_1(R)0, x_2(R)0$ as to (R) a, the optimal choice for u is $u=-0.5\,x_1(t)-0.414\,x_2(t)$

Illustrate the feed back control in a black diagram

(ix) An electrochemical system is governed by the equations $\frac{dx_1}{dt}=-x_1+u$ and $\frac{dx_2}{dt}+=x_1$ where u is a control variable so chosen that the cost functional $J=\int_0^{\alpha}\left(x_2^2+\frac{16}{3}u^2\right)dt$ is minimized.

Assumig the boundary conditions as in the previous problem, show that the optimal control u is given by $u(t)$ $=-0.366x_1(t)-0.443x_2(t)$

(x) In an inventory control production scheduting problem, the governing equation is $\frac{dI}{dt}=p$ where $I-I(t)$ is the invenvory level and $p=p(t)$ is the production rate remaining after the demand has been met. It is planned that over a fixed time interval $)<t<T$, I should be increased from its value 10 at $t=0$ by decreasing p in such a way that the cost functional

$$J=\int_0^T\left[(Q-I)^2+\alpha^2p^2\right] \text{ is minimized.}$$

Here, Q and a_2 are positive constants and $Q>10$. Determine the optimal production rate pond the inventory level.

(xi) The cost of a chemcial process is described by the functional $J = \int_0^1 \left(\frac{1}{2} y'^2 + y \right)$ where $y(0)=1$.

If $y(1)$ is not specified, using the transversatity conditions at $x = 1$, determine the optimum trajectory and the corresponding value of J and compare with the earlier result.

(xii) Describe the Pontryagin's Maximum Principle and illustrate it with the help of an example.

8. References :

Following tents are suggested for further study --

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(v) J. Macki & A. Strauss, Introduction to Optimal Control Theory, Springer, 1982.

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