Complex Functions and Properties

Sem- VI.

Paper- C13

Course- Mathematics(H) UG.

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Exponential function :

Let us consider the series $1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots$ to ∞, is called the exponential series.

This is a power series of the form $\sum_{n=0}^{\infty}\frac{z^n}{\lfloor n}$ where $a_n=\frac{1}{\lfloor n}$

ie $\sum_{n=0}^{\infty} a_n z^n$,

Now, $\frac{a_n}{a_{n+1}} = n+1 \to \infty$ as $n\to\infty$.

Thus, the radius of convergence of this series is infinite. Hence, the series is absolutely convergent for all finite values of z and it is a single valued continuous function of z. This function of z is called the exponential function of z and is denoted by e^z or $\exp(z)$.

$$\therefore\ e^z = \exp(z) = \sum_{n=0}^{\infty}\frac{z^n}{\lfloor n}$$

Let $z=x+iy$, then $\exp(z) = e^x(\cos y + i\sin y)$.

So, $u+iv$ be a non zero complex number and its polar representation be $r(\cos\theta+i\sin\theta)$. Since r is positive, $\log r$ is real and $r=e^{\log r}$

$$\therefore\ u+iv = e^{\log r}(\cos\theta+i\sin\theta)$$

$$= e^{\log r}.\, e^{i\theta} = \exp(\log r + i\theta)$$

Thus, when $u+iv$ is a given non zero complex number, there exists a complex number $z=\log r+i\theta$ s.t $\exp z = u+iv$.

So, the range of the exponential function of z is the entire complex plane excluding the origin.

Properties-1: Prove that $\exp(z_1)\exp(z_2) = \exp(z_1+z_2)$ where z_1 and z_2 are complex numbers.

Proof: Let $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$.

$$\therefore\ z_1+z_2 = x_1+x_2+i(y_1+y_2)$$

$$\exp(z_1) = e^{x_1}(\cos y_1 + i\sin y_1),\quad \exp(z_2) = e^{x_2}(\cos y_2 + i\sin y_2).$$

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$$\exp(z_1)\cdot\exp(z_2) = e^{x_1}(\cos y_1 + i\sin y_1)\cdot e^{x_2}(\cos y_2 + i\sin y_2)$$
$$= e^{x_1+x_2}\left[\cos(y_1+y_2) + i\sin(y_1+y_2)\right]$$
$$= \exp\left[(x_1+x_2) + i(y_1+y_2)\right]$$
$$= \exp(z_1+z_2).$$

<u>Property-2</u>: Prove that $\dfrac{\exp z_1}{\exp z_2} = \exp(z_1-z_2)$.

<u>Proof</u>: Since $\exp z_2$ is a non-zero complex number, $\dfrac{\exp z_1}{\exp z_2}$ is defined.

Let $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$

$z_1-z_2 = (x_1-x_2)+i(y_1-y_2)$.

$\exp z_1 = e^{x_1}(\cos y_1 + i\sin y_1)$, $\exp z_2 = e^{x_2}(\cos y_2 + i\sin y_2)$.

$$\frac{\exp z_1}{\exp z_2} = e^{x_1-x_2}\,\frac{\cos y_1 + i\sin y_1}{\cos y_2 + i\sin y_2}$$
$$= e^{x_1-x_2}\left[\cos(y_1-y_2) + i\sin(y_1-y_2)\right]$$
$$= \exp\left[(x_1-x_2) + i(y_1-y_2)\right]$$
$$= \exp(z_1-z_2).$$

<u>Property-3</u>: If n be an integer, then $(\exp z)^n = \exp(nz)$.

<u>Property-4</u>: If n be a fraction say $\frac{p}{q}$, $(\exp z)^n$ has q distinct values but $\exp(nz)$ is unique. In this case, $\exp(nz)$ is one of the values of $(\exp z)^n$.

<u>Property-5</u>: If n be an integer, then $\exp(z+2n\pi i) = \exp(z)$.

<u>Proof</u>: Since $e^{2n\pi i} = 1$, $n\in\mathbb{Z}$.

$\therefore \exp z\cdot 1 = \exp z\cdot\exp(2n\pi i) = \exp(z+2n\pi i)$

This shows that exponential function is periodic with period $2n\pi i$.

<u>Periodic function</u>: A complex function f is said to be a periodic function on its domain $D\subseteq\mathbb{C}$ if $\exists$ a non zero constant w such that for all integers, $f(z+nw) = f(z)$. holds $\forall\, z\in D$

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Note: $\exp(-z) = (\exp z)^{-1}$

$\boxed{\text{Ex}}$ Find all complex numbers z such that $\exp z = -1$.

Ans: Let $z = x+iy$.

Then $\exp z = -1 \Rightarrow e^x(\cos y + i \sin y) = -1$.

$\therefore \quad e^x \cos y = -1, \quad e^x \sin y = 0$.

We have, $e^x = 1$ and $\cos y = -1$ and $\sin y = 0$

$\Rightarrow y = (2n+1)\pi, \; n \in \mathbb{Z}$.

and $e^x = 1$

$\Rightarrow x = 0$.

Therefore, $z = (2n+1)\pi i$,

$\boxed{\text{Ex}}$ Find all complex numbers z such that $\exp(2z+1) = i$.

Ans: Let $z = x+iy$. Then $\exp(2z+1) = i$

$\Rightarrow \exp(2x+1+2iy) = i$

$\Rightarrow e^{2x+1}[\cos 2y + i \sin 2y] = i$

Therefore, $e^{2x+1} \cos 2y = 0, \quad e^{2x+1} \sin 2y = 1$.

We have $e^{2x+1} = 1$ and $\cos 2y = 0$, $\sin 2y = 1$.

Now, $e^{2x+1} = 1 \Rightarrow x = -\frac{1}{2}$.

$\cos 2y = 0$ and $\sin 2y = 1 \Rightarrow y = (4n+1)\frac{\pi}{4}$, where n is an integer. $\therefore \quad z = -\frac{1}{2} + (4n+1)\frac{\pi}{4} i$.

$\boxed{\text{Ex}}$ Solve : $\exp z = 1 + \sqrt{3} i$

Ans: Let $z = x+iy$. Then $\exp z = 1+\sqrt{3}i$

$\Rightarrow e^x(\cos y + i \sin y) = 1 + \sqrt{3} i$

$\therefore \quad e^x \cos y = 1$ and $e^x \sin y = \sqrt{3}$.

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$\because e^{2x} = 4 \Rightarrow e^x = 2 \quad [\because e^x > 0 \ \forall x \in \mathbb{R}]$.

$\therefore \cos y = \frac{1}{2}$ and $\sin y = \frac{\sqrt{3}}{2}$.

$\Rightarrow y = 2n\pi + \pi/3, \ n \in \mathbb{Z}$.

and $e^x = 2 \Rightarrow x = \log 2$.

Therefore, $z = \log 2 + (2n\pi + \pi/3)i, \ n \in \mathbb{Z}$.

$\boxed{\text{Ex}}$: If $\exp z$ is positive real number, prove that $\operatorname{Im} z = 2n\pi, n \in \mathbb{Z}$.

Ans: Let $\exp z = k$, k being positive real number.

Let $z = x + iy$.

$\therefore \exp z = k \Rightarrow e^x(\cos y + i\sin y) = k$

$\Rightarrow e^x \cos y = k$ and $e^x \sin y = 0$.

$\therefore e^{2x} = k^2$

$\Rightarrow e^x = k$, $k (>0) \in \mathbb{R}$.

Now, $e^x \cos y = k$ and $e^x \sin y = 0$

$\Rightarrow \cos y = 1$ and $\sin y = 0$.

$\Rightarrow y = 2n\pi, \ n \in \mathbb{Z}$.

$\therefore z = \log k + 2n\pi i, \ n \in \mathbb{Z}$.

$\operatorname{Im}(z) = 2n\pi$

$\boxed{\text{Ex}}$: If $\exp z$ is negative real number, prove that $\operatorname{Im} z = (2n+1)\pi, n \in \mathbb{Z}$.

Ans: Try yourself.

$\boxed{\text{Ex}}$: Find all complex numbers z satisfying $\exp(2z + \bar{z}) = 3 + 4i$

Ans: Let $z = x + iy$.

$\because \exp(2z + \bar{z}) = 3 + 4i$

$\Rightarrow \operatorname{Exp}[2(x+iy) + x - iy] = 3 + 4i$

$\Rightarrow \exp[3x + iy] = 3 + 4i$

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$\Rightarrow \quad e^{3x}\left[\cos y + i \sin y\right] = 3+4i$

$\Rightarrow \quad e^{3x}\cos y = 3$ and $e^{3x}\sin y = 4$.

$\therefore \quad (e^{3x})^2 = 25 \Rightarrow e^{3x} = 5$

$\Rightarrow x = \frac{1}{3}\log 5$

Also, $\cos y = \frac{3}{5}$ and $\sin y = \frac{4}{5}$.

$\therefore \quad y = 2n\pi + \tan^{-1}\frac{4}{3}, \; n \in \mathbb{Z}$.

$\therefore \quad z = \frac{1}{3}\log 5 + i\left(2n\pi + \tan^{-1}\frac{4}{3}\right), \; n \in \mathbb{Z}$.

Logarithmic function:

Let z be non-zero complex number. Then there exists a complex number w s.t. $\exp w = z$.

w is said to be an logarithmic of z.

Again, $\exp w = \exp(w + 2n\pi i), \; n \in \mathbb{Z}$.

$\therefore \exp(w + 2n\pi i) = z$ so, $w + 2n\pi i$ is also logarithmic of z.

$\therefore \quad \mathrm{Log}\, z = w + 2n\pi i$, 'logarithmic of z' is a many valued function of z. The principal logarithmic of z is obtained by putting a particular value of n. and denoted by $\log z$.

Since z is a non zero complex number, so the polar representation of z as $z = re^{i\theta}, \; -\pi < \theta \le \pi$.

$\therefore$ Let $w = u + iv$ be the logarithmic of z.

Then $\exp w = z$

$\Rightarrow \quad e^u(\cos v + i \sin v) = re^{i\theta}$

$\therefore \quad e^u = r$ and $v = \theta + 2n\pi$, $n \in \mathbb{Z}$.

$\therefore \quad \mathrm{Log}\, z = \log r + i(\theta + 2n\pi), \; -\pi < \theta \le \pi$.

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$$\therefore \quad \operatorname{Log} z = \log r + i(\theta + 2n\pi)$$
$$= \log|z| + i(\arg z + 2n\pi).$$

The principal logarithmic of z, denoted by $\log z$, corresponds to $n=0$.

$$\therefore \quad \log z = \log r + i\theta$$
$$= \log|z| + i \arg z.$$

$\boxed{\text{Ex}}$: Find $\operatorname{Log} z$ and $\log z$ where $z = 1 + i\tan\theta$, $\pi/2 < \theta < \pi$.

Ans: Let $z = r(\cos\varphi + i\sin\varphi)$ $\therefore$ $r\cos\varphi = 1$, $r\sin\varphi = \tan\theta$.

$$\therefore \quad r^2 = \sec^2\theta$$
$$\Rightarrow r = -\sec\theta \text{ as } \sec\theta < 0 \text{ for } \pi/2 < \theta < \pi.$$

So, $\cos\varphi = -\cos\theta$ and $\sin\varphi = -\sin\theta$

$\therefore$ $\varphi = \pi + \theta$, φ is not the principal argument of z.

$$\therefore \quad \arg z = \varphi - 2\pi = \pi + \theta - 2\pi$$
$$= \theta - \pi.$$

$$\therefore \quad \operatorname{Log} z = \log(-\sec\theta) + i(\theta - \pi + 2n\pi), \; n \in \mathbb{Z}.$$

and $\log z = \log(-\sec\theta) + i(\theta - \pi)$.

$\boxed{\text{Property-1}}$ If z_1 and z_2 be two distinct complex numbers such that $z_1, z_2 \neq 0$ Then prove that $\operatorname{Log} z_1 + \operatorname{Log} z_2 = \operatorname{Log}(z_1 \cdot z_2)$.

Ans: Since $z_1 \neq 0$, $z_2 \neq 0$.

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Then $z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

$$\operatorname{Log} z_1 = \log r_1 + i(\theta_1 + 2n\pi), \; n \in \mathbb{Z}.$$
$$\operatorname{Log} z_2 = \log r_2 + i(\theta_2 + 2m\pi), \; m \in \mathbb{Z}.$$
$$\operatorname{Log}(z_1 z_2) = \log(r_1 r_2) + i(\theta_1 + \theta_2 + 2p\pi), \; p \in \mathbb{Z}.$$
$$\operatorname{Log} z_1 + \operatorname{Log} z_2 = \log r_1 + \log r_2 + i(\theta_1 + \theta_2 + 2n\pi + 2m\pi).$$
$$= \log(r_1 r_2) + i(\theta_1 + \theta_2 + 2q\pi), \; q = m + n.$$

Since, p, q are arbitrary integers, $\operatorname{Log} z_1 + \operatorname{Log} z_2 = \operatorname{Log}(z_1 z_2)$.

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Note : If $z_1 = z_2$, $\operatorname{Log} z_1 + \operatorname{Log} z_2 = 2\log r_1 + i(2\theta_1 + 4n\pi)$, $n \in \mathbb{Z}$.

and $\operatorname{Log}(z_1 z_2) = 2\log r_1 + i(2\theta_1 + 2p\pi)$, $p \in \mathbb{Z}$.

The set of the general values of $\operatorname{Log} z_1 + \operatorname{Log} z_2$ is a proper subset of the set of the general values of $\operatorname{Log}(z_1 z_2)$.

Hence, $\operatorname{Log} z_1 + \operatorname{Log} z_2 \neq \operatorname{Log}(z_1 z_2)$ if $z_1 = z_2$.

Ex Prove that $\log z_1 + \log z_2 \neq \log z_1 z_2$.

Ans: Let $z_1 = i$, $z_2 = -1$, $z_1 z_2 = -i$

$|z_1| = 1$, $|z_2| = 1$, $|z_1 z_2| = 1$.

$\arg(z_1) = \frac{\pi}{2}$, $\arg(z_2) = \pi$, $\arg(z_1 z_2) = -\frac{\pi}{2}$

$\therefore \log z_1 = \log|z_1| + i\arg(z_1)$
$= i\frac{\pi}{2}$.

$\log z_2 = \log|z_2| + i\arg(z_2)$
$= i\pi$.

and $\log(z_1 z_2) = \log|z_1 z_2| + i\arg(z_1 z_2)$
$= -i\frac{\pi}{2}$.

Hence, $\log z_1 + \log z_2 \neq \log(z_1 z_2)$.

Property 2: If z_1 and z_2 be two distinct complex numbers such that $z_1 z_2 \neq 0$ then prove that $\operatorname{Log} z_1 - \operatorname{Log} z_2 = \operatorname{Log}\frac{z_1}{z_2}$.

Proof: Since $z_1 z_2 \neq 0$ so $z_1 \neq 0$, $z_2 \neq 0$.

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

$\therefore \frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$.

$\operatorname{Log} z_1 = \log r_1 + i(\theta_1 + 2n\pi)$, $n \in \mathbb{Z}$.

$\operatorname{Log} z_2 = \log r_2 + i(\theta_2 + 2m\pi)$, $m \in \mathbb{Z}$.

$\operatorname{Log}\frac{z_1}{z_2} = \log r_1 - \log r_2 + i(\theta_1 - \theta_2 + 2p\pi)$, $p \in \mathbb{Z}$.

$\operatorname{Log} z_1 - \operatorname{Log} z_2 = \log r_1 - \log r_2 + i(\theta_1 - \theta_2 + 2n\pi - 2m\pi)$.
$= \log r_1 - \log r_2 + i(\theta_1 - \theta_2 + 2q\pi)$, $q = n - m$.

Since, p and q are arbitrary integers,

$\operatorname{Log} z_1 - \operatorname{Log} z_2 = \operatorname{Log}\left(\frac{z_1}{z_2}\right)$.

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Note: (i) If $z_1 = z_2$, $\operatorname{Log} z_1 - \operatorname{Log} z_2 = 0$

and $\operatorname{Log}\frac{z_1}{z_2} = \operatorname{Log} 1 = 2n\pi i$, $n \in \mathbb{Z}$.

$\therefore$ $\operatorname{Log} z_1 - \operatorname{Log} z_2 \neq \operatorname{Log}\frac{z_1}{z_2}$.

(ii) Prove that $\log z_1 - \log z_2 \neq \log\frac{z_1}{z_2}$.

Ans. Let $z_1 = -1$, $z_2 = -i$. Then $\frac{z_1}{z_2} = -i$

and $|z_1| = |z_2| = 1$, $\arg(z_1) = \pi$, $\arg(z_2) = -\frac{\pi}{2}$

$\arg\left(\frac{z_1}{z_2}\right) = -\frac{\pi}{2}$.

$\therefore$ $\log z_1 = \pi i$, $\log z_2 = -\frac{\pi}{2} i$, $\log\left(\frac{z_1}{z_2}\right) = -\frac{\pi}{2} i$.

Hence, $\log z_1 - \log z_2 = \frac{3\pi i}{2} \neq \log\frac{z_1}{z_2}$.

$\boxed{\text{Property-3}}$: If $z \neq 0$ and m be a positive integer, then prove that $\operatorname{Log} z^m \neq m \operatorname{Log} z$.

Proof: Let $z = r(\cos\theta + i\sin\theta)$

Then $z^m = r^m(\cos m\theta + i\sin m\theta)$ [By De-Moivre's theorem].

$\operatorname{Log} z = \log r + i(\theta + 2n\pi)$, $n \in \mathbb{Z}$.

$\operatorname{Log} z^m = \log r^m + i(m\theta + 2p\pi)$, $p \in \mathbb{Z}$

$\therefore$ $m \operatorname{Log} z = m\log r + i(m\theta + 2mn\pi)$

$= \log r^m + i(m\theta + 2q\pi)$, $q = mn$.

Since p is arbitrary, and q is a multiple of m, each value of $m\operatorname{Log} z$ is a value of $\operatorname{Log} z^m$ but not conversely.

So, the set of values of $m\operatorname{Log} z$ is a proper sub set of the set of values of $\operatorname{Log} z^m$. Therefore, $\operatorname{Log} z^m \neq m\operatorname{Log} z$.

For example, let $z = i$, $m = 2$.

$2\operatorname{Log} z = 2\operatorname{Log} i = (4n+1)\pi i$, $n \in \mathbb{Z}$.

$\operatorname{Log} z^2 = \operatorname{Log}(-1) = (2k+1)\pi i$, $k \in \mathbb{Z}$.

Each value of $2\operatorname{Log} i$ is a value of $\operatorname{Log} i^2$ but not conversely.

$\therefore$ $\operatorname{Log} i^2 \neq 2\operatorname{Log} i$.

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$\boxed{\text{Ex}}$: If x is real, prove that $i \log \frac{x-i}{x+1} = \pi - 2\tan^{-1}x, x>0$
$= -\pi - 2\tan^{-1}x, x \le 0.$

Ans: Let $x>0$,
Let $x+i = r(\cos\theta + i\sin\theta)$, $0<\theta<\pi/2$.
Then $x = r\cos\theta$, $1 = r\sin\theta$ and $\cot\theta = x$.

Now, $\log \frac{x-i}{x+i} = \log \frac{r(\cos\theta - i\sin\theta)}{r(\cos\theta + i\sin\theta)} = \log(e^{-2i\theta})$
$= \log[\cos(-2\theta) + i\sin(-2\theta)]$.

$0<\theta<\frac{\pi}{2} \Rightarrow -\pi < -2\theta < 0 \Rightarrow -2\theta$ is the principal argument.

Therefore, $i \log \frac{x-i}{x+i} = i(-2\theta)i = 2\theta$. —(1)

$\cot\theta = x \Rightarrow \tan(\pi/2 - \theta) = x$

$0<\theta<\frac{\pi}{2} \Rightarrow 0 < \frac{\pi}{2} - \theta < \frac{\pi}{2}$ and $\tan(\frac{\pi}{2} - \theta) = x$

$\Rightarrow \frac{\pi}{2} - \theta = \tan^{-1}x$

From (1), $i \log \frac{x-i}{x+i} = 2\theta$
$= \pi - 2\tan^{-1}x$.

Let ~~case~~ $x<0$

Let $x+i = r(\cos\theta + i\sin\theta)$, $\frac{\pi}{2} < \theta < \pi$. Then $\cot\theta = x$.

$\log \frac{x-i}{x+i} = \log[\cos(-2\theta) + i\sin(-2\theta)]$

Now, $\frac{\pi}{2} < \theta < \pi \Rightarrow -2\pi < -2\theta < -\pi \Rightarrow 0 < -2\theta + 2\pi < \pi$
$\Rightarrow -2\theta + 2\pi$ is the principal argument.

Therefore, $i \log \frac{x-i}{x+i} = i(-2\theta + 2\pi)i$
$= 2\theta - 2\pi$ —(2)

$\therefore \cot\theta = x \Rightarrow \tan(\pi/2 - \theta) = x$, $\frac{\pi}{2} < \theta < \pi \Rightarrow -\pi < -\theta < -\pi/2$
$\Rightarrow -\pi/2 < \frac{\pi}{2} - \theta < 0$

$\therefore \tan(\pi/2 - \theta) = x \Rightarrow \pi/2 - \theta = \tan^{-1}x$.

From (2), $i \log \frac{x-i}{x+1} = -2(\pi - \theta)$
$= -\pi - 2\tan^{-1}x$.

Let $x = 0$.
$i \log \frac{x-i}{x+i} = i \log(-1) = i(\pi i) = -\pi = -\pi - 2\tan^{-1}x$.

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Therefore, $i \log\frac{x-i}{x+i} = \pi - 2\tan^{-1}x$ if $x>0$

$= -\pi - 2\tan^{-1}x$ if $x \le 0$.

~~Trigonometric function~~

$\boxed{\text{Ex}}$ Show that $\cos\left[i \log\frac{a-ib}{a+ib}\right] = \frac{a^2-b^2}{a^2+b^2}$, a, b are real numbers.

Ans: Let $(a+ib) = r(\cos\theta + i\sin\theta)$. Where $(a,b) \neq (0,0)$, $-\pi < \theta \le \pi$

∴ $a = r\cos\theta$, $b = r\sin\theta$, $\tan\theta = \frac{b}{a}$, $-\pi < \theta \le \pi$.

$$\therefore \frac{a-ib}{a+ib} = \frac{r(\cos\theta - i\sin\theta)}{r(\cos\theta + i\sin\theta)} = \frac{e^{-i\theta}}{e^{i\theta}} = e^{-2i\theta}$$

$$= e^{-2i\theta + 2k\pi i}, \quad k \in \mathbb{Z}.$$

where $-\pi < -2\theta + 2k\pi \le \pi$.

$$\cos\left[i\log\frac{a-ib}{a+ib}\right] = \cos[i(-2i\theta + 2k\pi i)]$$

$$= \cos(2\theta - 2k\pi)$$

$$= \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\frac{b^2}{a^2}}{1+\frac{b^2}{a^2}} = \frac{a^2-b^2}{a^2+b^2}.$$

$$\therefore \cos\left[i\log\frac{a-ib}{a+ib}\right] = \frac{a^2-b^2}{a^2+b^2}.$$

$\boxed{\text{Ex}}$: Prove that $\sin\left[i\log\frac{a-ib}{a+ib}\right] = \frac{2ab}{a^2+b^2}$ $a, b \in \mathbb{R}$.

Ans: Try yourself.

$\boxed{\text{Ex}}$: Prove that $\tan\left[i\log\frac{x-iy}{x+iy}\right] = 2$ represents a rectangular hyperbola.

Ans: Here $(x,y) \neq (0,0)$ as $\frac{x-iy}{x+iy}$ is not defined.

Let $x+iy = r(\cos\theta + i\sin\theta)$, $-\pi < \theta \le \pi$.

∴ $x = r\cos\theta$, $y = r\sin\theta$. and $\tan\theta = \frac{y}{x}$, $-\pi < \theta \le \pi$.

$$\frac{x-iy}{x+iy} = \frac{r(\cos\theta - i\sin\theta)}{r(\cos\theta + i\sin\theta)} = e^{-2i\theta} = e^{-2i\theta + 2k\pi i}, \quad k \in \mathbb{Z}.$$

$$, \quad -\pi < -2\theta + 2k\pi \le \pi.$$

$$\therefore \tan\left[i\log\frac{x-iy}{x+iy}\right] = \tan[i(-2\theta + 2k\pi)i]$$

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$\Rightarrow \tan(2\theta - 2k\pi)$

$= \tan 2\theta$.

Now, $\tan\left[i \log \frac{x-iy}{x+iy}\right] = 2$

$\Rightarrow \tan 2\theta = 2$

$\Rightarrow \dfrac{2\tan^2\theta}{1-\tan^2\theta} = 2$

$\Rightarrow \dfrac{2(y/x)^2}{1-(y/x)^2} = 2$

$\Rightarrow x^2 - y^2 = xy$, which is rectangular hyperbola.

<u>Defn</u>: Let a and z are complex numbers where $a \neq 0$, then $\text{Log}_a z$ is defined as $\text{Log}_a z = \dfrac{\text{Log}\, z}{\text{Log}\, a}$.

$\boxed{\text{Ex}}$ Prove that $\text{Log}_i(-1) = \dfrac{(2n+1)\pi}{(4m+1)\frac{\pi}{2}}$, m, n being integers.

<u>Ans</u>: $\text{Log}_i(-1) = \dfrac{\text{Log}(-1)}{\text{Log}\, i} = \dfrac{\log 1 + i(2n\pi + \pi)}{\log|i| + i\left(2m\pi + \frac{\pi}{2}\right)}$

$= \dfrac{(2n+1)\pi}{(4m+1)\frac{\pi}{2}}$.

<u>Trigonometric functions</u>

The series $1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$ to ∞ and $z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$ to ∞ are convergent for all finite value of z.

So, $\cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \cdots$

$\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots$

$e^{iz} = 1 + \dfrac{iz}{\lfloor 1} + \dfrac{i^2 z^2}{\lfloor 2} + \dfrac{(iz)^3}{\lfloor 3} + \dfrac{(iz)^4}{\lfloor 4} + \cdots$

$= \left(1 - \dfrac{z^2}{\lfloor 2} + \dfrac{z^4}{\lfloor 4} - \cdots\right) + i\left(z - \dfrac{z^3}{\lfloor 3} + \dfrac{z^5}{\lfloor 5} - \cdots\right)$

$= \cos z + i \sin z$.

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$\therefore\ e^{-iz} = \cos z - i\sin z.$

So $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ and $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

Where z is real or complex.

Properties: $\boxed{Ex}$ When z is a complex number, prove that $\sin^2 z + \cos^2 z = 1$.

Ans. According to def^n $\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}(t + \frac{1}{t})$

$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{2i}(t - \frac{1}{t})$

$\therefore \sin^2 z + \cos^2 z = \frac{1}{4}\left[(t+\frac{1}{t})^2 - (t - \frac{1}{t})^2\right]$ where $t = \exp(iz) = e^{iz}$.

$= \frac{1}{4} \cdot 4t \cdot \frac{1}{t} = 1.$

$\boxed{Ex}$: If z_1, z_2 be complex number then prove that

(i) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

(ii) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

Proof: (i) $\sin(z_1 + z_2) = \dfrac{\exp[(z_1+z_2)i] - \exp[-i(z_1+z_2)]}{2i}$

$$= \frac{\exp(iz_1)\exp(iz_2) - \exp(-iz_1)\exp(-iz_2)}{2i}$$

$$= \frac{t_1 t_2 - \frac{1}{t_1 t_2}}{2i}, \quad t_1 = \exp(iz_1),\ t_2 = \exp(iz_2).$$

$$= \frac{t_1^2 t_2^2 - 1}{2i t_1 t_2}$$

$$= \frac{(t_1^2 - 1)(t_2^2 + 1) + (t_1^2 + 1)(t_2^2 - 1)}{4i t_1 t_2}$$

$$= \frac{(t_1 - \frac{1}{t_1})}{2i}\frac{(t_2 + \frac{1}{t_2})}{2} + \frac{(t_1 + \frac{1}{t_1})}{2}\cdot\frac{(t_2 - \frac{1}{t_2})}{2i}$$

$$= \sin z_1 \cos z_2 + \cos z_1 \sin z_2.$$

(ii) $\cos(z_1 + z_2) = \dfrac{\exp[i(z_1+z_2)] + \exp[-i(z_1+z_2)]}{2}$

$$= \frac{\exp(iz_1)\exp(iz_2) + \exp(-iz_1)\exp(-iz_2)}{2}$$

$$= \frac{t_1 t_2 + \frac{1}{t_1 t_2}}{2}, \quad t_1 = \exp(iz_1),\ t_2 = \exp(iz_2)$$

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$$= \frac{t_1^2 t_2^2 + 1}{2 t_1 t_2}$$

$$= \frac{(t_1^2+1)(t_2^2+1) + (t_1^2-1)(t_2^2-1)}{4 t_1 t_2}$$

$$= \frac{(t_1 + \frac{1}{t_1})}{2} \frac{(t_2 + \frac{1}{t_2})}{2} - \frac{(t_1 - \frac{1}{t_1})(t_2 - \frac{1}{t_2})}{2i}$$

$$= \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

$\boxed{Ex}$: If x, y are real

(i) $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$.

(ii) $\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$.

Ans: (i) $$\sin(x+iy) = \frac{\exp[i(x+iy)] - \exp[-i(x+iy)]}{2i}$$

$$= \frac{e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)}{2i}$$

$$= \sin x \cdot \frac{e^{y} + e^{-y}}{2} - \cos x \cdot \frac{e^{y} - e^{-y}}{2i}$$

$$= \sin x \cosh y + i \cos x \sinh y.$$

(ii) $$\cos(x+iy) = \frac{\exp[i(x+iy)] + \exp[-i(x+iy)]}{2}$$

$$= \frac{e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x)}{2}$$

$$= \cos x \frac{e^{y} + e^{-y}}{2} - i \sin x \frac{e^{y} - e^{-y}}{2}$$

$$= \cos x \cosh x - i \sin x \sinh y.$$

Hyperbolic functions:

Let z be complex, the hyperbolic functions of $\cosh z$, $\sinh z$ are defined as $\cosh z = \frac{\exp z + \exp(-z)}{2}$ $\sinh z = \frac{\exp z - \exp(-z)}{2}$

Properties: (i) $\cosh^2 z - \sinh^2 z = 1$. (ii) $\cos 2z = \cosh^2 z - \sinh^2 z$.

(iii) If z_1, z_2 be complex numbers then

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$$

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2.$$

Proof: $$\sinh(z_1 + z_2) = \frac{\exp(z_1 + z_2) - \exp(-z_1 - z_2)}{2}$$

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$$= \frac{\exp z_1 \exp z_2 - \exp(-z_1)\exp(-z_2)}{2}$$

$$= \frac{t_1t_2 - \frac{1}{t_1t_2}}{2}, \quad t_1 = \exp(z_1), \quad t_2 = \exp(z_2).$$

$$= \frac{t_1^2t_2^2 - 1}{2t_1t_2}$$

$$= \frac{(t_1^2-1)(t_2^2+1) + (t_1^2+1)(t_2^2-1)}{4t_1t_2}$$

$$= \frac{(t_1 - \frac{1}{t_1})}{2}\frac{(t_2 + \frac{1}{t_2})}{2} + \frac{(t_1 + \frac{1}{t_1})}{2} \cdot \frac{(t_2 - \frac{1}{t_2})}{2}$$

$$= \sinh z_1 \cdot \cosh z_2 + \cosh z_1 \sinh z_2.$$

$\boxed{\text{Ex}}$ Find all values of z such that $\cos z = 0$.

Ans: Let $z = x+iy$ $\therefore \cos(x+iy) = 0$

$\Rightarrow \cos x \cosh y - i\sin x \sinh y = 0.$

$\Rightarrow \cos x \cosh y = 0$ (1) and $\sin x \sinh y = 0$ (2)

From (1) $\cos x = 0$, $\cosh y \neq 0$

$\Rightarrow x = (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$.

From (2) $\sinh y = 0$, $\therefore \sin(2n+1)\frac{\pi}{2} \neq 0$.

$\Rightarrow y = 0.$

$\therefore z = (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$.

$\boxed{\text{Ex}}$: Find the general solution of $\sin z = 2i$.

Ans: Since $\sin z = 2i$

$$\Rightarrow \frac{\exp(iz) - \exp(-iz)}{2i} = 2i$$

$$\Rightarrow t - \frac{1}{t} = -4 \quad [\because t = \exp(iz)].$$

$$\Rightarrow t^2 + 4t - 1 = 0$$

$$\Rightarrow t = -2 \pm \sqrt{5}.$$

$$\therefore \exp(iz) = -2 \pm \sqrt{5}.$$

Let $\exp(iz) = -2+\sqrt{5}$

$$iz = \operatorname{Log}(-2+\sqrt{5})$$

$$= \log|-2+\sqrt{5}| + i(2n\pi), \quad n \in \mathbb{Z}.$$

$$= \log\sqrt{(-2+\sqrt{5})^2} + 2n\pi i \quad \text{—(1)}$$

$z = 2n\pi + i \log(2+\sqrt{5})$.

Also, $\exp(iz) = (2-\sqrt{5})$

$\Rightarrow iz = \text{Log}[-2-\sqrt{5}]$

$= \log(2+\sqrt{5}) + (2n+1)\pi i, \quad n \in \mathbb{Z}$.

$\therefore z = (2n+1)\pi - i \log(2+\sqrt{5})$. (2)

So combining (1) and (2) we have

$z = n\pi + (-1)^n \log(2+\sqrt{5})$.

$\boxed{\text{Ex}}$: Find the general solution of $\cos z = -2$

Ans: Since $\cos z = -2$.

$$\frac{\exp(iz) + \exp(-iz)}{2} = -2.$$

$\Rightarrow \left(t + \frac{1}{t}\right) = -4 \quad [t = \exp(iz)]$

$\Rightarrow t^2 + 4t + 1 = 0$

$\Rightarrow t = -2 \pm \sqrt{3}$

i.e. $\exp(iz) = -2 \pm \sqrt{3}$

Let $\exp(iz) = -2+\sqrt{3}$

$\Rightarrow iz = \text{Log}(-2+\sqrt{3})$

$= \text{Log}\{(-\sqrt{3}+2)(-1)\}$

$= \log(2-\sqrt{3}) + (2n+1)\pi i, \quad n \in \mathbb{Z}$.

$\therefore z = (2n+1)\pi - i \log(2-\sqrt{3})$ — (1)

Also $\exp(iz) = -2-\sqrt{3}$

$\Rightarrow iz = \text{Log}[(2+\sqrt{3})(-1)]$

$= \log(2+\sqrt{3}) + (2n+1)\pi i, \quad n \in \mathbb{Z}$.

$\therefore z = (2n+1)\pi + i \log(2-\sqrt{3})$. — (2)

Combining (1) & (2) we have,

$z = (2n+1)\pi \pm i \log(2-\sqrt{3}) \quad n \in \mathbb{Z}$.

$\boxed{\text{Ex}}$: Find the general solution of $\sin z = 2$.

Ans: Given that $\sin z = 2$

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$\Rightarrow t - \frac{1}{t} = 4i \quad [\because t = \exp(iz)]$

$\Rightarrow t^2 - 4it - 1 = 0$

$\Rightarrow t = (2 \pm \sqrt{3})i$

When $t = (2+\sqrt{3})i$, then $iz = \mathrm{Log}(2+\sqrt{3})i$

$= \log(2+\sqrt{3}) + i\,(2n\pi + \pi/2),\ n \in \mathbb{Z}.$

$\therefore \quad z = 2n\pi + \frac{\pi}{2} - i\log(2+\sqrt{3})$

When $t = (2-\sqrt{3})i$, then $iz = \mathrm{Log}(2-\sqrt{3})i$

$= \log(2-\sqrt{3}) + (2n\pi + \pi/2)i$

$z = 2n\pi + \pi/2 - i\log(2-\sqrt{3})$

$= (2n+1)\pi - \{\pi/2 - i\log(2+\sqrt{3})\}$

Combining the above, we have, $z = n\pi + (-1)^n\{\frac{\pi}{2} - i\log(2+\sqrt{3})\},\ n \in \mathbb{Z}.$

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