Chapter 5

Solution of System of Linear Equations

A system of m linear equations in n unknowns (variables) is written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots &\quad \cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &= b_i \\ \cdots\cdots\cdots\cdots\cdots\cdots &\quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned} \tag{5.1}$$

The quantities $x_1, x_2, \ldots, x_n$ are the **unknowns (variables)** of the system and $a_{11}, a_{12}, \ldots, a_{mn}$ are the **coefficients** of the unknowns of the system. The numbers $b_1, b_2, \ldots, b_m$ are **constant** or **free terms** of the system.

The above system of equations (5.1) can be written as

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad i = 1, 2, \ldots, m. \tag{5.2}$$

Also, the system of equations (5.1) can be written in matrix form as

$$\mathbf{AX} = \mathbf{b}, \tag{5.3}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{bmatrix} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_m \end{bmatrix}. \tag{5.4}$$

The system of linear equation (5.1) is **consistent** if it has a solution. If a system of linear equations has no solution, then it is **inconsistent** (or **incompatible**). A consistent system of linear equations may have one solution or several solutions and is said to be **determinate** if there is one solution and **indeterminate** if there are more than one solution.

Generally, the following three types of the **elementary transformations** to a system of linear equations are used.

Interchange: The order of two equations can be changed.

Scaling: Multiplication of both sides of an equation of the system by any non-zero number.

Replacement: Addition to (subtraction from) both sides of one equation of the corresponding sides of another equation multiplied by any number.

A system in which the constant terms $b_1, b_2, \ldots, b_m$ are zero is called a **homogeneous** system.

Two basic techniques are used to solve a system of linear equations:

(i) direct method, and (ii) iteration method.

Several direct methods are used to solve a system of equations, among them following are most useful.

(i) Cramer's rule, (ii) matrix inversion, (iii) Gauss elimination, (iv) decomposition, etc.

The most widely used iteration methods are (i) Jacobi's iteration, (ii) Gauss-Seidal's iteration, etc.

Direct Methods

5.1 Cramer's Rule

To solve a system of linear equations, a simple method (but, not efficient) was discovered by Gabriel Cramer in 1750.

Let the determinant of the coefficients of the system (5.2) be $D = |a_{ij}|; i, j = 1, 2, \ldots, n$, i.e., $D = |A|$. In this method, it is assumed that $D \neq 0$. The Cramer's rule is described in the following. From the properties of determinant

$$x_1 D = x_1 \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} x_1 a_{11} & a_{12} & \cdots & a_{1n} \\ x_1 a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_1 a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & a_{12} & \cdots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$ [Using the operation $C_1' = C_1 + x_2C_2 + \cdots + x_nC_n$.]

$$= \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix} \text{[Using (5.1)]}$$

$$= D_{x_1}(say).$$

Therefore, $x_1 = D_{x_1}/D$.

Similarly, $x_2 = \frac{D_{x_2}}{D}, \ldots, x_n = \frac{D_{x_n}}{D}$.

In general, $x_i = \frac{D_{x_i}}{D}$, where

$$D_{x_i} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1\,i-1} & b_1 & a_{1\,i+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2\,i-1} & b_2 & a_{2\,i+1} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{n\,i-1} & b_n & a_{n\,i+1} & \cdots & a_{nn} \end{vmatrix},$$

$= 1, 2, \cdots, n$

xample 5.1.1 Use Cramer's rule to solve the following systems of equations

$$x_1 + x_2 + x_3 = 2$$
$$2x_1 + x_2 - x_3 = 5$$
$$x_1 + 3x_2 + 2x_3 = 5.$$

olution. The determinant D of the system is

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & 3 & 2 \end{vmatrix} = 5.$$

he determinants D_1, D_2 and D_3 are shown below:

$$D_1 = \begin{vmatrix} 2 & 1 & 1 \\ 5 & 1 & -1 \\ 5 & 3 & 2 \end{vmatrix} = 5, \quad D_2 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & -1 \\ 1 & 5 & 2 \end{vmatrix} = 10, \quad D_3 = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 5 \\ 1 & 3 & 5 \end{vmatrix} = -5.$$

hus, $x_1 = \frac{D_1}{D} = \frac{5}{5} = 1, x_2 = \frac{D_2}{D} = \frac{10}{5} = 2, x_3 = \frac{D_3}{D} = -\frac{5}{5} = -1.$

herefore the solution is $x_1 = 1, x_2 = 2, x_3 = -1$.

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5.5 Gauss Elimination Method

In this method, the variables are eliminated by a process of systematic elimination. Suppose the system has n variables and n equations of the form (5.1). This procedure reduces the system of linear equations to an equivalent upper triangular system which can be solved by back–substitution. To convert an upper triangular system, x_1 is eliminated from second equation to nth equation, x_2 is eliminated from third equation to nth equation, x_3 is eliminated from fourth equation to nth equation, and so on and finally, x_{n-1} is eliminated from nth equation.

To eliminate x_1, from second, third, $\cdots$, and nth equations the first equation is multiplied by $-\frac{a_{21}}{a_{11}}, -\frac{a_{31}}{a_{11}}, \ldots, -\frac{a_{n1}}{a_{11}}$ respectively and successively added with the second, third, $\cdots$, nth equations (assuming that $a_{11} \neq 0$). This gives

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \cdots + a_{2n}^{(1)}x_n &= b_2^{(1)} \\
a_{32}^{(1)}x_2 + a_{33}^{(1)}x_3 + \cdots + a_{3n}^{(1)}x_n &= b_3^{(1)} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots \\
a_{n2}^{(1)}x_2 + a_{n3}^{(1)}x_3 + \cdots + a_{nn}^{(1)}x_n &= b_n^{(1)},
\end{aligned} \tag{5.12}$$

where

$$a_{ij}^{(1)} = a_{ij} - \frac{a_{i1}}{a_{11}}a_{1j}; \quad i, j = 2, 3, \ldots, n.$$

Again, to eliminate x_2 from the third, forth, ..., and nth equations the second equation is multiplied by $-\frac{a_{32}^{(1)}}{a_{22}^{(1)}}, -\frac{a_{42}^{(1)}}{a_{22}^{(1)}}, \ldots, -\frac{a_{n2}^{(1)}}{a_{22}^{(1)}}$ respectively (assuming that $a_{22}^{(1)} \neq 0$), and successively added to the third, fourth, ..., and nth equations to get the new system of equations as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \cdots + a_{2n}^{(1)}x_n &= b_2^{(1)} \\ a_{33}^{(2)}x_3 + \cdots + a_{3n}^{(2)}x_n &= b_3^{(2)} \\ \cdots\cdots\cdots\cdots\cdots\cdots &\cdots \\ a_{n3}^{(2)}x_3 + \cdots + a_{nn}^{(2)}x_n &= b_n^{(2)}, \end{aligned} \tag{5.13}$$

where

$$a_{ij}^{(2)} = a_{ij}^{(1)} - \frac{a_{i2}^{(1)}}{a_{22}^{(1)}} a_{2j}^{(1)}; \quad i,j = 3,4,\ldots,n.$$

Finally, after eliminating x_{n-1}, the above system of equations become

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \cdots + a_{2n}^{(1)}x_n &= b_2^{(1)} \\ a_{33}^{(2)}x_3 + \cdots + a_{3n}^{(2)}x_n &= b_3^{(2)} \\ \cdots\cdots\cdots\cdots &\cdots \\ a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}, \end{aligned} \tag{5.14}$$

where,

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} a_{kj}^{(k-1)};$$

$i,j = k+1,\ldots,n;\ \ k = 1,2,\ldots,n-1$, and $a_{pq}^{(0)} = a_{pq};\ \ p,q = 1,2,\ldots,n$.

Now, by back substitution, the values of the variables can be found as follows:

From last equation we have, $x_n = \dfrac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$, from the last but one equation, i.e., $(n-1)$th equation, one can find the value of x_{n-1} and so on. Finally, from the first equation we obtain the value of x_1.

The evaluation of the elements $a_{ij}^{(k)}$'s is a **forward substitution** and the determination of the values of the variables x_i's is a **back substitution** since we first determine the value of the last variable x_n.

Note 5.5.1 The method described above assumes that the diagonal elements are non-zero. If they are zero or nearly zero then the above simple method is not applicable to solve a linear system though it may have a solution. If any diagonal element is zero or very small then partial pivoting should be used to get a solution or a better solution.

It is mentioned earlier that if the system is diagonally dominant or real symmetric and positive definite then no pivoting is necessary.

Example 5.5.1 Solve the equations by Gauss elimination method.
$2x_1 + x_2 + x_3 = 4,\ \ x_1 - x_2 + 2x_3 = 2,\ \ 2x_1 + 2x_2 - x_3 = 3.$

Solution. Multiplying the second and third equations by 2 and 1 respectively and subtracting them from first equation we get

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 4 \\ 3x_2 - 3x_3 &= 0 \\ -x_2 + 2x_3 &= 1. \end{aligned}$$

Multiplying third equation by -3 and subtracting from second equation we obtain

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 4 \\ 3x_2 - 3x_3 &= 0 \\ 3x_3 &= 3. \end{aligned}$$

From the third equation $x_3 = 1$, from the second equations $x_2 = x_3 = 1$ and from the first equation $2x_1 = 4 - x_2 - x_3 = 2$ or, $x_1 = 1$.
Therefore the solution is $x_1 = 1, x_2 = 1, x_3 = 1$.

Example 5.5.2 Solve the following system of equations by Gauss elimination method (use partial pivoting).

$$\begin{aligned} x_2 + 2x_3 &= 5 \\ x_1 + 2x_2 + 4x_3 &= 11 \\ -3x_1 + x_2 - 5x_3 &= -12. \end{aligned}$$

Solution. The largest element (the pivot) in the coefficients of the variable x_1 is -3, attained at the third equation. So we interchange first and third equations

$$\begin{aligned} -3x_1 + x_2 - 5x_3 &= -12 \\ x_1 + 2x_2 + 4x_3 &= 11 \\ x_2 + 2x_3 &= 5. \end{aligned}$$

Multiplying the second equation by 3 and adding with the first equation we get,

$$\begin{aligned} -3x_1 + x_2 - 5x_3 &= -12 \\ x_2 + x_3 &= 3 \\ x_2 + 2x_3 &= 5 \end{aligned}$$

The second pivot is 1, which is at the positions a_{22} and a_{32}. Taking $a_{22} = 1$ as pivot to avoid interchange of rows. Now, subtracting the third equation from second equation, we obtain

$$\begin{aligned} -3x_1 + x_2 - 5x_3 &= -12 \\ x_2 + x_3 &= 3 \\ -x_3 &= -2. \end{aligned}$$

Now by back substitution, the values of x_3, x_2, x_1 are obtained as

$x_3 = 2,\ x_2 = 3 - x_3 = 1,\ x_1 = -\dfrac{1}{3}(-12 - x_2 + 5x_3) = 1.$

Hence the solution is $x_1 = 1$, $x_2 = 1$, $x_3 = 2$.