## Motion of circular cylinder :

### B.C. (Boundary condiⁿ for cylindrical motion):

Here we consider two dimensional irrotational motion produced by the motion of cylinders in an infinite mass of liquid at rest at infinity or when a cylinder is inserted in a stream

The stream function $\psi$ is determine in the light of following condition :

(i) $\psi$ satisfies Laplace equation i.e, $\nabla^2\psi = 0$ at every point of the liquid

(ii) The liquid is at rest at infinity so that $\frac{\partial \psi}{\partial x} = 0$ and $\frac{\partial \psi}{\partial y} = 0$ at $\infty$

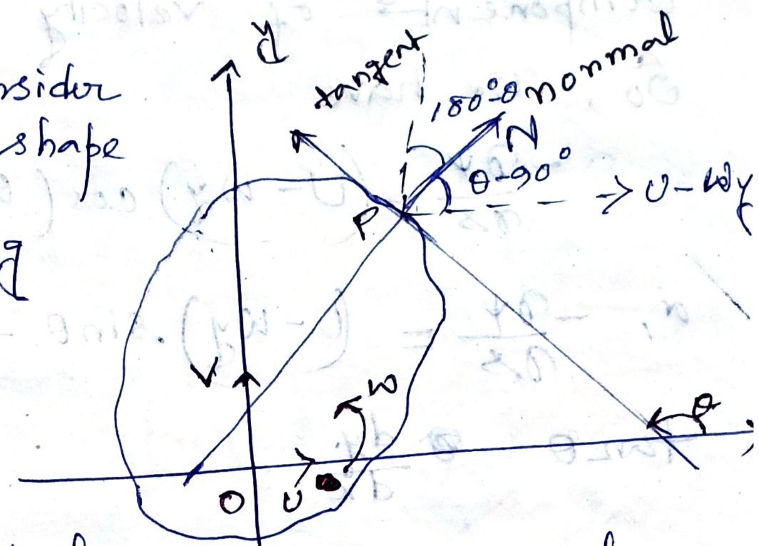
(iii) Along any fixed boundary, the normal component of velocity must be zero so that $-\frac{\partial \psi}{\partial s} = 0$ and hence $\psi$ = constant

that means the boundary must coincide with the stream line $\psi$ = con

(iv) Along the boundary of moving cylinder, the normal component of the velocity of the liquid must be equal to normal component of the velocity of the cylinder

General motion:

Let us consider a cylinder of any shape and size be moving in a fluid at rest at infinity.




Let the cylinder be moving perpendicular to its generators and let the reference axes be fixed in the cylinder which are moving with linear velocities U and V resp. and rotation velocity $\omega$ as shown in figure.

We consider any point $P(x, y)$ on the boundary and the tangent PT at P makes an angle $\theta$ with $x$ axis. and PN is normal.

Then we have, $x = r\cos\theta$

$y = r\sin\theta$

i.e, $\dot{x} = \frac{dx}{dt} = -r\sin\theta\,\dot{\theta} = -\omega y$

$\dot{y} = \frac{dy}{dt} = r\cos\theta\,\dot{\theta} = \omega x$

So, the velocity component at P along x axis and y axis are $U + \dot{x}$ and $V + \dot{y}$

i.e, $U - \omega y$ and $V + \omega x$

But the normal components of velocity of boundary is equal to the normal components of velocity of liquid (by Condⁿ (iv))

So, we have

$$-\frac{\partial \psi}{\partial s} = (U - \omega y)\cos(\theta - 90^\circ) + (V + \omega x)\cos(\ldots)$$

or, $$-\frac{\partial \psi}{\partial s} = (U - \omega y)\sin\theta - (V + \omega x)\cos\theta$$

$$\tan\theta = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{ds} = \sin\theta$$

$$\frac{dx}{ds} = \cos\theta$$

$$\therefore -\frac{\partial \psi}{\partial s} = (U - \omega y)\frac{dy}{ds} - (V + \omega x)\frac{dx}{ds}$$

or, $$-\frac{\partial \psi}{\partial s}\, ds = (U - \omega y)\, dy - (V + \omega x)\, dx$$

$$\Rightarrow -d\psi = (U - \omega y)\, dy - (V + \omega x)\, dx$$

Integrating both sides we have

$$-\psi = Uy - \frac{\omega y^2}{2} - Vx - \frac{\omega x^2}{2} + c$$

or, $$\psi = (Vx - Uy) + \frac{\omega}{2}(x^2 + y^2) + c$$

, where $c$ is arbitrary integrating constant

This is the required expression for the general motion of the cylinder.

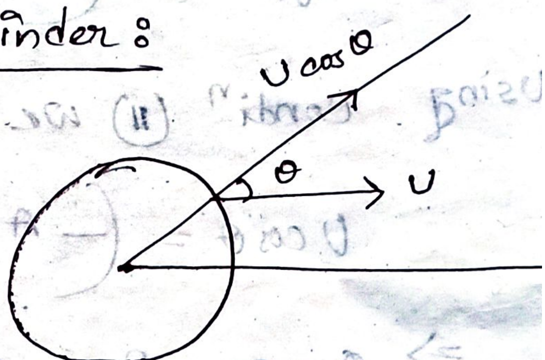
Case 1: If there is pure rotational motion then we have $U=0$, $V=0$ and hence,

$$\psi = \frac{\omega}{2}(x^2+y^2) + C$$

Case 2: If the cylinder has only linear velocity, along $x$ axis then $V=0$, $\omega=0$

So, $\psi = -Uy + C$

## Motion of the circular cylinder:




Let the circular cylinder be moving with velocity $U$ along $x$ axis in an infinite mass of liquid at rest at infinity.

The velocity potential $\phi$ must satisfy the Laplace eq$^n$

$$\nabla^2\phi = 0 \quad \text{——— (1)}$$

The solu$^n$ of this eq$^n$, in polar form can be written as

i) $r^{\pm n}\cos n\theta$, $r^{\pm n}\sin n\theta$

ii) $$\left(-\frac{\partial\phi}{\partial r}\right)_{r=a} = U\cos\theta$$

(iii) The liquid is at rest at infinity

i.e, $\left(-\frac{\partial \phi}{\partial r}\right)_{r=\infty} = 0$ and $\left(-\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)_{r=\infty} = 0$

To satisfy these condition, let us suppose velocity potential in the form of

$$\phi = \left(Ar + \frac{B}{r}\right)\cos\theta \quad \text{——— (2)}$$

, where A, B are arbitrary constants

Now, $-\frac{\partial \phi}{\partial r} = \left(-A + \frac{B}{r^2}\right)\cos\theta$ ——— (3)

Using condi$^n$ (ii) we have

$$U\cos\theta = \left(-A + \frac{B}{a^2}\right)\cos\theta$$

$\Rightarrow A = 0, \quad B = a^2U$ (?)

By condi$^n$ (iii)

$$\therefore \boxed{\phi = \frac{Ua^2}{r}\cos\theta} \quad \text{——— (4)}$$

But we know that there is a $\varphi \to \psi$ relation

$$\frac{\partial \varphi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad (\text{C-R eqn})$$

or, $-\frac{Ua^2}{r^2}\cos\theta = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$

$\Rightarrow \frac{\partial \psi}{\partial \theta} = -\frac{Ua^2}{r}\cos\theta$

Integrating, $\psi = -\frac{Ua^2}{r}\sin\theta$, [neglecting integrating consta

- Hence the complex potential of the fluid motion

$$w = \varphi + i\psi$$

$$= \frac{Ua^2}{r}\cos\theta - i\frac{Ua^2}{r}\sin\theta$$

$$= \frac{Ua^2}{r}(\cos\theta - i\sin\theta)$$

$$\boxed{w = \frac{Ua^2}{r}e^{-i\theta}}$$

$$= \frac{Ua^2}{re^{i\theta}} = \frac{Ua^2}{z}$$

, this is the complex potential for circular cylinder motion

The stream lines are given by $\psi = \text{constant}$

i.e., $-\frac{Ua^2}{r}\sin\theta = \text{Constant} = k$, say

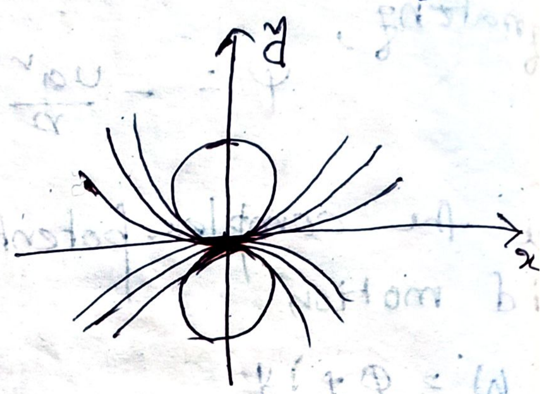
or, $kr^2 = -Ua^2 r\sin\theta$

or, $r^2 = k' r\sin\theta$.

, where $k' = -\frac{Ua^2}{k}$ = con

or, $\boxed{x^2+y^2-k'y=0}$.

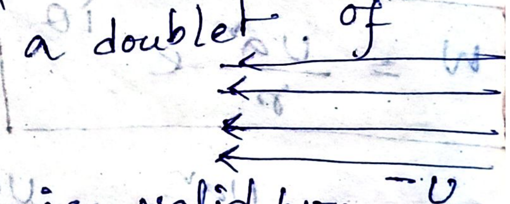
This represents a system of circle whose centre is at $(0, k'/2)$ with radius $\sqrt{\frac{k'^2}{4}}$ i.e, the system of circle all touching $x$ axis at origin





Liquid streaming past a fixed cylinder :

A uniform flow with a velocity $U$ in the direction of increasing $x$ axis is suppose on the flow due to a doublet of strength $\mu$.





Now, the superposition is valid here because the governing equation for either the velocity potential of the stream function is linear.

Consider the superposition of a uniform rectilinear flow at the origin. Then the complex potential for the resulting flow is of the form is

$W = Uz + \frac{\mu}{z}$

Therefore $\phi + i\psi = Ur e^{i\theta} + \frac{\mu}{r} e^{-i\theta}$

, where $z = r e^{i\theta}$

So equating both sides, we have

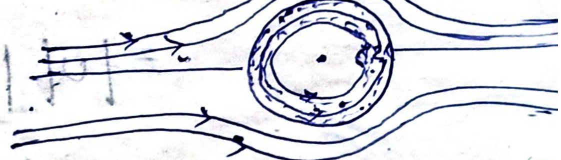
~~$\phi = Ur\cos\theta + \mu i\theta$~~

$$\phi = \left(Ur + \frac{\mu}{r}\right)\cos\theta$$

$$\psi = \left(Ur - \frac{\mu}{r}\right)\sin\theta$$

For general values of $\mu$ the stream funⁿ $\psi$ is clearly variable, but if we take strength of the doublet to be $\mu = Ua^2$ $(r=a)$

Then, $\psi = 0$. which gives the surface of the circle and then flow pattern is as follows



Here the flow field due to dublet is shown by doted lines. and we notice that the doublet flow is entirely contained within the circle while uniform flow is deflected by the doublet in such a manner that it is entirely outside of the circle. The circle itself is common ~~common~~ to the two flow fields.

Thus the flow field due to a doublet of strength $Ua^2$ and the uniform rectilinear flow of magnitude $U$ gives the same flow as that for a uniform flow of magnitude $U$ passed a circular cylinder of radius $a$

Hence the complex potential for a

uniform flow of magnitude U ~~passed~~ past a ~~circular~~ circular cylinder of radius a is

$$W = U\left(z + \frac{a^2}{z}\right) \qquad (1)$$

The velocity distribution $\vec{q}$ on a point of the circular cylinder is

$$|\vec{q}| = \left|\frac{dW}{dz}\right| = |U|\left|1 - \frac{a^2}{z^2}\right|$$

$$= |U|\left|1 - e^{-2i\theta}\right|$$

$$= |U|\left|1 - (\cos 2\theta - i\sin 2\theta)\right|$$

$$= |U|\left|(1 - \cos 2\theta) + i\sin 2\theta\right|$$

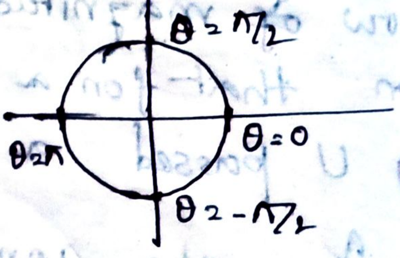
$$= |U|\left|2\sin^2\theta + 2i\sin\theta\cos\theta\right|$$

$$= |U|\,|2\sin\theta|\,|\sin\theta + i\cos\theta|$$

$$= |U|\,|2\sin\theta|$$

this shows that fluid velocity ~~potential~~ vanishes at $\theta = 0$ or $\theta = \pi$ and these points are called stagnation points of the flow.

The maximum value of the velocity of the surface of the cylinder can be determine for $\theta = \pm\frac{\pi}{2}$ and



Hence, $q_{max} = |\vec{q}|_{\theta = \pm \pi/2}$

$= 2|U| = 2 \times$ free stream velocity.

Let $p$ be the pressure at any point on the boundary of the cylinder.

Then by Bernoulli's equation, we have

$$\frac{p}{\rho} + \frac{1}{2}|\vec{q}|^2 = \text{constant} = A$$

Since $p = \Pi$, $|\vec{q}| = U$

$$\therefore \quad A = \frac{\Pi}{\rho} + \frac{1}{2}U^2$$

Hence,

$$\frac{p}{\rho} + \frac{1}{2}|\vec{q}|^2 = \frac{\Pi}{\rho} + \frac{1}{2}U^2$$

$$\text{or, } p - \Pi = \frac{\rho}{2}\left(U^2 - |\vec{q}|^2\right)$$

$$= \frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right)$$

The liquid will remain in contant with the boundary of the circular cylinder so long as the pressure at every point of it is positive. and it follows that the liquid cannot sustained a negetive pressure. If the pressure be negetive then cavitation will occure which is opposite to our discussion.

The formation of a vacous space in fluid is called cavitation.

Therefore for pressure to be positive at every point i.e, $p > 0$

i.e, $\Pi + \frac{1}{2}\rho U^2(1-4\sin^2\theta) > 0$.

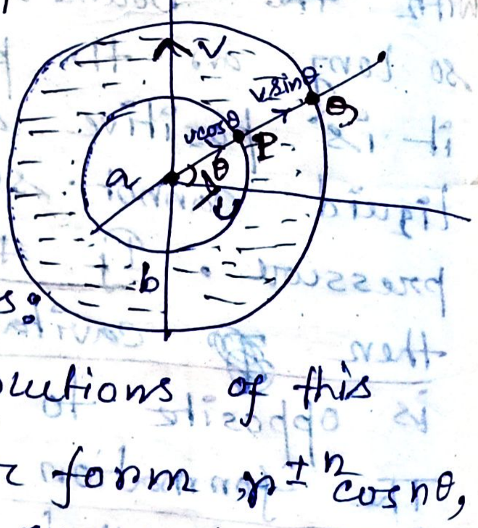
or, $\Pi - \frac{3}{2}\rho U^2 > 0$ as $\theta = \pm \pi/2$

and i.e; when $U^2 < \frac{2\Pi}{3\rho}$

If $U$ exceeds the above value then liquid will cavitate at the sides of the cylinder.

## Initial motion between two co-axial cylinder:

Determine the velocity potential and stream funn at any point of a fluid contained between co-axial cylinder of radii $a$ and $b$ $(a<b)$ when the cylinders are moved suddenly parallel to themselves in direction at right angle with velocities $U$ and $V$ resp.




$\Rightarrow$ The velocity potential $\phi$ must satisfy the following conditions:

i) $\nabla^2\phi = 0$ the solutions of this equation are in polar form $r^{\pm n}\cos n\theta$, $r^{\pm n}\sin n\theta$ for every positive integer $n$

ii) $\left(-\frac{\partial\varphi}{\partial r}\right)_{r=a} = U\cos\theta$ and.

$$\left(-\frac{\partial\varphi}{\partial r}\right)_{r=b} = V\sin\theta.$$

To satisfy above conditions, let us assume that velocity potential may be in the form of

$$\varphi = \left(Ar + \frac{B}{r}\right)\cos\theta + \left(Cr + \frac{D}{r}\right)\sin\theta \qquad (1)$$

$$\therefore \frac{\partial\varphi}{\partial r} = \left(A - \frac{B}{r^2}\right)\cos\theta + \left(C - \frac{D}{r^2}\right)\sin\theta.$$

find A, B, C, D using ii)

Using ii) at $r = a$,

$$\left(-\frac{\partial\varphi}{\partial r}\right)_{r=a} = U\cos\theta.$$

or, $\left(-A + \frac{B}{a^2}\right)\cos\theta + \left(-C + \frac{D}{a^2}\right)\sin\theta = U\cos\theta$

Equating co-efficient both sides,

$$-A + \frac{B}{a^2} = U$$

$$-C + \frac{D}{a^2} = 0$$

Also, $\left(-A + \frac{B}{b^2}\right)\cos\theta + \left(-C + \frac{D}{b^2}\right)\sin\theta = V\sin\theta$

$$-A + \frac{B}{b^2} = 0$$

$$-C + \frac{D}{b^2} = V.$$

$\therefore A = \dfrac{Ua^2}{b^2-a^2}$

$B = \dfrac{Ua^2b^2}{b^2-a^2}$

$C = \dfrac{-Vb^2}{b^2-a^2}$

$D = \dfrac{-Va^2b^2}{b^2-a^2}$

$D = a^2 C$

$= \dfrac{-a^2 V b^2}{b^2-a^2}$

$$\therefore \phi = -\left[\frac{Ua^2}{b^2-a^2} r + \frac{Ua^2b^2}{(b^2-a^2)r}\right]\cos\theta + \left[-\frac{Vb^2}{b^2-a^2} r + \frac{Va^2b^2}{(b^2-a^2)r}\right]\sin\theta$$

$$= \frac{-Ua^2}{a^2-b^2}\left(r+\frac{b^2}{r}\right)\cos\theta + \frac{Vb^2}{a^2-b^2}\left(r+\frac{a^2}{r}\right)\sin\theta \quad (2)$$

By Cauchy-Riemann eqns we have

$$\frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$\Rightarrow \frac{\partial \psi}{\partial \theta} = r\left[-\frac{Ua^2}{a^2-b^2}\left(1-\frac{b^2}{r^2}\right)\cos\theta + \frac{Vb^2}{a^2-b^2}\left(1-\frac{a^2}{r^2}\right)\sin\theta\right]$$

$$= \frac{Ua^2}{b^2-a^2}\left(r-\frac{b^2}{r}\right)\cos\theta - \frac{Vb^2}{b^2-a^2}\left(r-\frac{a^2}{r}\right)\sin\theta$$

integrating w.r.t. $\theta$,

$$\psi = \frac{Ua^2}{b^2-a^2}\left(r-\frac{b^2}{r}\right)\sin\theta + \frac{Vb^2}{b^2-a^2}\left(r-\frac{a^2}{r}\right)\cos\theta \quad (3)$$

Ignoring integrating constant,

Thus the values of $\phi$ and $\psi$, hold only at the instant when the cylinders are on starting.

## Kinetic Energy:

The kinetic energy of an infinite mass of liquid moving irrotationally is, given by,

$$T = \int_V \frac{1}{2}\rho |\vec{q}|^2 dV$$

$$= \frac{1}{2}\rho \int_V |\vec{q}|^2 dV$$

For irrotational motion, we have

$$\vec{\nabla} \times \vec{q} = \vec{0}$$

$$\Rightarrow \vec{q} = -\vec{\nabla}\phi$$

So, $T = \frac{1}{2}\rho \int_V \vec{\nabla}\phi \cdot \vec{\nabla}\phi \, dv$

$$= \frac{1}{2}\rho \int_S \left( -\phi\nabla^2\phi - \phi\frac{\partial\phi}{\partial n} \right) ds$$

[by Green's theorem]

$$= -\frac{1}{2}\rho \int_S \phi \frac{\partial\phi}{\partial n} ds$$

[because $\nabla^2\phi = 0$]

But $\frac{\partial\phi}{\partial n}$ = normal derivative of $\phi$

$$\therefore \frac{\partial\varphi}{\partial n} = \vec{\nabla}\varphi \cdot \hat{n}$$

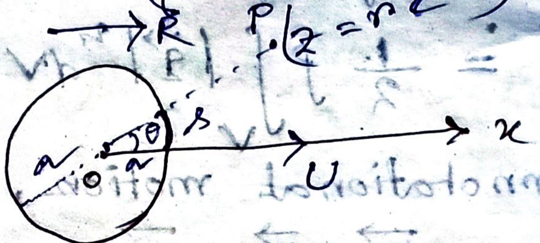
Hence $T = -\frac{1}{2}\iint_S \varphi(\vec{\nabla}\varphi \cdot \hat{n})\,ds$.

This measures the kinetic energy of the closed surfaces of liquid outside the surfaces.

## Equations of motion of a circular cylinder :

For the simple case, we take the origin at the centre of the cylinder and the motion is along $x$ axis.

Let us suppose that circular cylinder of radius $a$ is moving in a velocity $U$ along $x$ axis in an infinite mass of liquid at rest at $\infty$.



Then the complex potential for the resulting motion is given by

$$W = \frac{Ua^2}{z}$$

$$= \frac{Ua^2}{r} e^{-i\theta}$$

$$\Rightarrow \varphi + i\psi = \frac{Ua^2}{r}(\cos\theta - i\sin\theta)$$

Comparing with real parts we have,

$$\phi = \frac{Ua^2}{r}\cos\theta$$

So, $\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = \left[-\frac{Ua^2}{r^2}\cos\theta\right]_{r=a}$

$$= -U\cos\theta$$

Let $T_1$ be the K.E. of the liquid on the boundary of the cylinder and $T_2$ is that of the cylinder.

Let $\sigma$ and $\rho$ be the densities of the cylinder and the liquid resp.

Then we have,

$$T_1 = \left[-\frac{\rho}{2}\int_S \phi\frac{\partial\phi}{\partial n}\,ds\right]_{r=a}$$

Here $\frac{\partial\phi}{\partial n}$ denotes derivative of $\phi$ along the radius vector.

For the circular cylinder,

$|z| = a$ and $s = a\theta$

$\therefore ds = a\,d\theta$

Hence $T_1 = -\frac{\rho}{2}\int_0^{2\pi}\left[\phi\frac{\partial\phi}{\partial r}\right]_{r=a} a\,d\theta$

[ $\frac{\partial\phi}{\partial n}$ is normal along the radius ]

$$= -\frac{\rho}{2}\int_0^{2\pi}(-Ua\cos\theta \; U\cos\theta)\, a\, d\theta$$

$$= \frac{\rho U^2 a^2}{4}\int_0^{2\pi} 2\cos^2\theta \, d\theta$$

$$= \frac{\rho U^2 a^2}{4}\int_0^{2\pi}(1+\cos 2\theta)\, d\theta$$

$$= \frac{\rho U^2 a^2}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{\rho U^2 a^2}{4}\left[2\pi + \frac{\sin 4\pi}{2}\right]$$

$$= \frac{\pi \rho U^2 a^2}{2} = \frac{1}{2}\pi a^2 \rho U^2$$

$$= \frac{1}{2} M' U^2, \text{ say}$$

, where $M'$ = mass of the liquid displaced by the cylinder

$$= \pi a^2 \rho$$

Also, $\left[T_2\right]$ = $\left[\text{K.E. of the cylinder}\right.$

$$\left. \right] = \frac{1}{2} M U^2$$

Thus the total K.E. of the cylinder

and Liquid $= T_1 + T_2$

$$= \frac{1}{2}(M' + M)U^2$$

If R denotes the force parallel to the direction of motion

Then By the Law of Conservation of energy

i.e, ~~change in K.E.~~ is the Rate of change of total energy the rate at which work is being done by external forces at the boundary

~~i.e, the rate of change of~~

i.e, $\frac{d}{dt}(K.E) = \frac{\text{work done}}{\text{time}}$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}(M+M')U^2\right) = \frac{\text{force} \times \text{displacement}}{\text{time}}$$

$$= \text{force} \times \text{velocity}$$

$$= RU$$

$$\Rightarrow \cancel{\tfrac{2}{2}}(M+M')U\dot{U} = RU$$

$$\Rightarrow M\dot{U} = \cancel{(R-M'\dot{U})}\ R - M'\dot{U}$$

This shows that the presence of liquid offers resistance (drag force) of ammount $M'\dot{U}$ in the direction of motion of the cylinder.

If $\frac{R'}{M}$ is a constant conservative external force per unit mass acting on the cylinder and liquid alike, then the energy equation at time $t$ provides $\cancel{\frac{1}{2}(M+M')U^2}$ $\cancel{(M-M')\left(\frac{R'}{M}\right)}$

$$\frac{1}{2}(M+M')U^2-(M-M')\left(\frac{R'}{M}\right)x=\text{constant}$$

, where $x$ is distance moved by the cylinder in the direction of $R'$.

∴ By Differentiating we have

$$(M+M')U\dot{U}-(M-M')\left(\frac{R'}{M}\right)\frac{dx}{dt}=0$$

$$\Rightarrow (M+M')U\dot{U}-(M-M')\frac{R'}{M}U=0$$

$$\left(\because \frac{dx}{dt}=U\right)$$

$$\text{or, } M\dot{U}=\cancel{(M-M')R}\left(\frac{M-M'}{M+M'}\right)R'$$

$$\dot{U}=\left(\frac{\pi a^2\cdot 1\cdot \sigma-\pi a^2\cdot \rho}{\pi a^2\sigma+\pi a^2\rho}\right)R'M$$

$$\text{or, } M\dot{U}=\left(\frac{\sigma-\rho}{\sigma+\rho}\right)R'$$

Let $U = u + iv$ and $R' = X' + iY'$

Then we get two cartesian equation of motion,

$$M\dot{u} = \left(\frac{\sigma - \rho}{\sigma + \rho}\right) X'$$

and $$M\dot{v} = \left(\frac{\sigma - \rho}{\sigma + \rho}\right) Y'$$

Above equations follows that the effect of the presence of the liquid is to reduce external forces in the ratio $(\sigma - \rho) : (\sigma + \rho)$

Q: Calculate circulation about circular cylinder. [If k be a constant circulation around a circular cylinder of radius a, then we have

$\text{cir} = \int \vec{q} \cdot d\vec{r}$

$$k = \left( \frac{1}{r} \frac{\partial \varphi}{\partial \theta} 2\pi r \right)$$

i.e, $$-\frac{k}{2\pi} = \frac{\partial \varphi}{\partial \theta}$$

or, $$k = \left(-2\pi \frac{\partial \varphi}{\partial \theta}\right)$$

Integrating, $$\varphi = -\frac{k}{2\pi}\theta.$$

But $$\frac{\partial \varphi}{\partial r} = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

and, $$\frac{1}{r}\frac{\partial \varphi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

$$\psi = \frac{k}{2\pi}\log r.$$

$$W = \varphi + i\psi$$

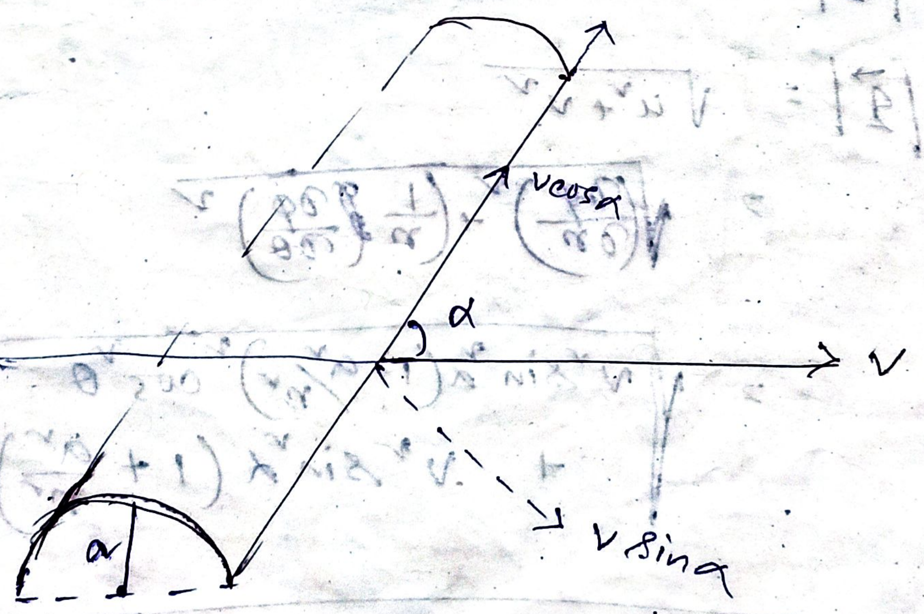
$$= \frac{ik}{2\pi}(\log z) \quad [\because z = re^{i\theta}]$$

or, $\frac{iK}{2\pi} \log z = W$.

or, $K = \frac{2\pi}{i} \frac{W}{\log z}$

Q. A stream of water of a great depth is flowing with uniform velocity $V$ over a plane level bottom. An infinite cylinder, of which the cross section is a semicircle of radius $a$, lie on its flat side with its generating lines making an angle $\alpha$ with the undisturbed stream lines. Prove that the resultant fluid pressure per unit length ~~of~~ on the curved surface is $2a\Pi - \frac{5}{3}\rho a V^2 \sin^2\alpha$, where $\Pi$ is the ~~at~~ fluid pressure at a great distance from the cylinder.

$\Rightarrow$



Let $V$ be the velocity of the stream over a plane level. Then the components of velocity are $v\cos\alpha$ acts along the generator and $v\sin\alpha$ acts $\perp^r$ to the generator of the cylinder.

Here $V\cos\alpha$ doesnot exert any pressure on the cylinder.

So, only the components $V\sin\alpha$ will give rise a velocity potential $\phi$ s.t.

$$\phi = V\sin\alpha\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$\therefore \frac{\partial\phi}{\partial r} = V\sin\alpha\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$\frac{\partial\phi}{\partial\theta} = -V\sin\alpha\left(r + \frac{a^2}{r}\right)\sin\theta$$

Let $\vec{q}$ be the velocity at any point $(r,$

~~So, $|\vec{q}|^2 = \sqrt{u^2 +}$~~

So, $|\vec{q}| = \sqrt{u^2 + v^2}$

$$= \sqrt{\left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2}$$

$$= \sqrt{V^2\sin^2\alpha\left(1 - \frac{a^2}{r^2}\right)^2\cos^2\theta + V^2\sin^2\alpha\left(1 + \frac{a^2}{r^2}\right)^2 \sin^2\theta}$$

$$= \sqrt{V^2\sin^2\alpha\left[1 - \frac{2a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4}\right]} \quad \text{---(1)}$$

Now, the pressure $p$ at any point $(r, \theta)$ is given by Bernoulli's eqn,

$$\frac{p}{\rho} + \frac{1}{2}|\vec{q}|^2 = \text{Constant} = C, \text{ say.}$$

$$\therefore \frac{p}{\rho} = C - \frac{1}{2}|\vec{q}|^2$$

$$= C - \frac{1}{2} V^2 \sin^2\alpha \left[1 - \frac{2a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4}\right] \quad \text{——— (2)}$$

But when $p = \Pi$, $r = \infty$

then $C = \frac{\Pi}{\rho} + \frac{1}{2}V^2 \sin^2 a$

Hence $\frac{p}{\rho} = \frac{\Pi}{\rho} + \frac{V^2\sin^2\alpha}{2}\left(\frac{2a^2}{r^2}\cos 2\theta - \frac{a^4}{r^4}\right)$

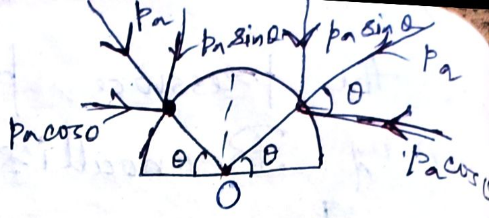
Let $p_a$ be the pressure at the point $(a, \theta)$ on the curved surface of the cylinder and which is given by,

$$p_a = \Pi + \frac{V^2\sin^2\alpha\, \rho}{2}\left(2\cos 2\theta - 1\right)$$

$$= \Pi + \frac{1}{2}V^2\sin^2\alpha\, \rho \left[4\cos^2\theta - 2 - 1\right]$$

$$= \Pi + \frac{1}{2}\rho V^2 \sin^2\alpha\left(4\cos^2\theta - 3\right)$$

$$= \Pi - \frac{3}{2}\rho V^2\sin^2\alpha + 2\rho V^2\sin^2\alpha\cos^2\theta \quad \text{——— (3)}$$




The component $p_a \cos\theta$ ~~neutrilizes~~ neutralizes each other, as the pressure exerted on the curved surface of the semi circular cylinder, is equal

Thus the total resultant pressure on the semi circular cylinder is per unit length is given by

$$P = \int_0^{\pi} p_a \sin\theta \, a \, d\theta$$

$$= 2a\int_0^{\pi}\left(\Pi - \frac{3}{2}\rho v^2 \sin^2\alpha + 2\rho v^2 \sin^2\alpha \cos^2\theta\right)\sin\theta \, d\theta$$

$$= \cancel{a\left[\Pi\cos\theta + \frac{3}{2}\rho v^2 \sin^2\alpha\cos\theta\right]_0^{\pi}}$$

$$= a\left(\Pi - \frac{3}{2}\rho v^2 \sin^2\alpha\right)2 + 2\rho a v^2 \sin^2\alpha \int_0^{\pi} - \cos^2\theta \, d(\cos\theta)$$

3π - cosπ)
-1-1

$$= 2a\left(\Pi - \frac{3}{2}\rho v^2 \sin^2\alpha\right) + -2\rho a v^2 \sin^2\alpha \cdot \left(\frac{\cos 3\theta}{3}\right)_0^{\pi}$$

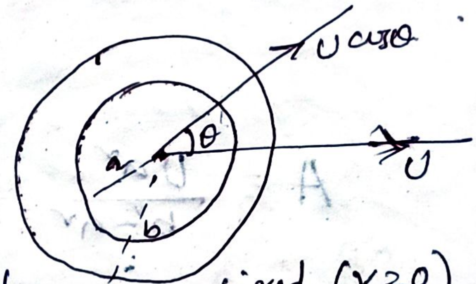
$$= 2a\left(\pi - \frac{3}{2}\rho v^2 \sin^2\alpha\right) + \frac{4}{3}\rho a v^2 \sin^2\alpha$$

$$= 2a\pi - \frac{5}{3}\rho a v^2 \sin^2\alpha$$

**Ex!** The space between two infinitely long co-axial cylinder of radii $a$ and $b$ resp. is filled with homogenous liquid of density $\rho$ and the inner cylinder suddenly moves with velocity $U$ $\perp^r$ to the axes, the other one being kept fixed. Show that the resultant impulsive pressure on a length $l$ of the inner cylinder is $\pi\rho a^2 l\left(\frac{b^2+a^2}{b^2-a^2}\right)U$.

$\Rightarrow$

Let the common region be filled by the liquid of density $\rho$.




Since, the inner cylinder suddenly moves with velocity $U$ $\perp^r$ to the axis and the outer cylinder being kept fixed, so the velocity potential $\phi$ must satisfy Laplace eq$^n$ with boundary condition

$$\left.\begin{aligned}\left(-\frac{\partial\phi}{\partial r}\right)_{r=a} &= U\cos\theta \\ \left(-\frac{\partial\phi}{\partial r}\right)_{r=b} &= 0\end{aligned}\right\}$$

Let the form of $\varphi$ be

$$\varphi = \left(Ar + \frac{B}{r}\right)\cos\theta + \left(Cr + \frac{D}{r}\right)\sin\theta$$

, where $A, B, C, D$ are arbitrary constants

$$\frac{\partial\varphi}{\partial r} = \left(A - \frac{B}{r^2}\right)\cos\theta + \left(C - \frac{D}{r^2}\right)\sin\theta$$

Using boundary conditions, we have

$$A = \frac{Ua^2}{b^2-a^2}, \quad B = \frac{Ua^2b^2}{b^2-a^2}, \quad C = D = 0.$$

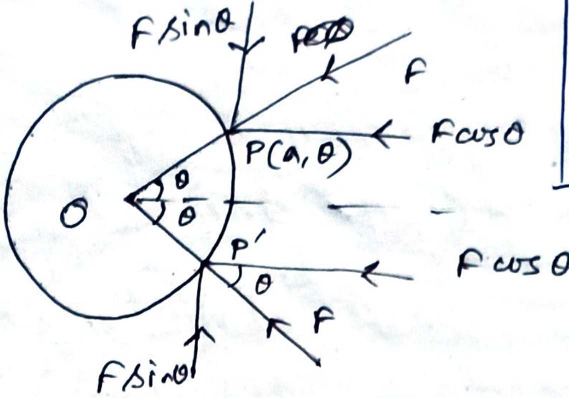
and hence,

$$\varphi = \left(\frac{Ua^2r}{b^2-a^2} + \frac{Ua^2b^2}{(b^2-a^2)r}\right)\cos\theta$$

$$= \frac{Ua^2}{b^2-a^2}\left(r + \frac{b^2}{r}\right)\cos\theta \quad (1)$$

Let $F$ be the impulsive pressure at the point $P(a,\theta)$ on the inner cylinder.

$Im$ = force × time
$= mf \times t$
$= \frac{m(v-u)}{t} \times t$
$= mv - mu$




So, $F = (\rho\phi)_{r=a}$

$$= \frac{\rho a U}{\cancel{b^2-a^2}}\left(\frac{a^2+b^2}{b^2-a^2}\right)\cos\theta.$$

Hence the total impulsive pressure on the cylinder of the length $l$ is given by

$$= \int_0^{2\pi} F\cos\theta\, l(a\,d\theta)$$

$$= \frac{(a^2+b^2)\rho a U l}{b^2-a^2}\int_0^{2\pi}\cos^2\theta\, a\,d\theta \qquad 1+\cos 2\theta$$

$$= \rho a^2 l U\left(\frac{a^2+b^2}{b^2-a^2}\right)\frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \rho a^2 L U\left(\frac{a^2+b^2}{b^2-a^2}\right)\frac{1}{2}\,2\pi$$

$$= \rho a^2 L U \pi\left(\frac{a^2+b^2}{b^2-a^2}\right)$$