LINEAR TRANSFORMATION

Theorem :- Let V and W be two vector spaces and let $T: V \to W$ be linear. If $\{v_1, v_2, \ldots, v_n\}$ be the basis of V then $\text{span}\{T(v_1), T(v_2), \ldots, T(v_n)\} = R(T)$; $[R(T) = \{T(x) \in W : x \in V\}]$

Proof :-

$$T(v_i) \in R(T), \; i = 1, 2, \ldots, n$$

$$\text{Span}\{T(v_1), T(v_2), \ldots, T(v_n)\} \subseteq R(T)$$

Now, $w \in R(T) \in W$ such that $T(v) = w, \; v \in V$

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

$$\therefore w = T(v) = T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$$

$$= c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n) \quad [\because T \text{ is linear}]$$

$$\therefore w \in \text{Span}\{T(v_1), T(v_2), \ldots, T(v_n)\}$$

$$\therefore R(T) \subseteq \text{Span}\{T(v_1), T(v_2), \ldots, T(v_n)\}$$

$$\therefore R(T) = \text{Span}\{T(v_1), T(v_2), \ldots, T(v_n)\}$$

Rank dimension theorem or Sylvester theorem :- Let $T: V \to W$ be a linear transformation. V and W be two vector spaces and V be finite dimensional. Then

$$\dim(N(T)) + \dim(R(T)) = \dim V$$

Proof :- Let $\{x_1, x_2, \ldots, x_m\}$ be the basis of $N(T)$.

i.e., $\dim(N(T)) = m$.

Now, $\{x_1, x_2, \ldots, x_m\}$ is L.I set

Now we can extend the L.I set $\{x_1, x_2, \ldots, x_m\}$ to the basis of V, let it be $\{x_1, x_2, \ldots, x_m, v_1, v_2, \ldots, v_n\}$

$$\therefore \dim V = m + n.$$

We have to show $\{T(v_1), T(v_2), \ldots T(v_n)\}$ be basis of $R(T)$.

To show L.I,

$$c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n) = 0$$

$$\Rightarrow T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = 0 \quad [\because T \text{ is linear mapping}]$$

$$\therefore c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \in N(T)$$

$$\therefore c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = d_1 x_1 + d_2 x_2 + \cdots + d_m x_m$$

$$\Rightarrow a_1 x_1 + a_2 x_2 + \cdots + a_m x_m + c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

, where $a_i = -d_i; \; i = 1(1)m$

$\therefore a_i = 0, c_i = 0$.
$; i = 1(1)m, \quad i = 1(1)n$. $\left[\because \{x_1, x_2, \ldots, x_m, v_1, v_2, \ldots, v_n\} \text{ is basis of } V\right]$

$\therefore$ The set $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is L.I.

Let ~~ω =~~ $\omega \in R(T)$ s.t. $\omega = T(v)$, $v \in V$.

$\therefore v = a_1x_1 + a_2x_2 + \cdots + a_mx_m + c_1v_1 + c_2v_2 + \cdots + c_nv_n$

$$\therefore \omega = T(v) = a_1T(x_1) + a_2T(x_2) + \cdots + a_mT(x_m) + c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n)$$

$$= c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n)$$

$[\because x_1, x_2, \ldots, x_m$ are elements of $N(T)]$

$\therefore \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is basis of $R(T)$.

$\therefore \dim R(T) = n$.

Hence, $\dim(N(T)) + \dim(R(T)) = m + n = \dim V$.

Hence, the result is proved.

H/W

Theorem :- Let V and W be two vector spaces and $T: V \to W$ be a linear transformation. Then T is one-one if and only if $\ker(T) = \{0\}$.

Theorem :- Let V and W be two finite ~~dimen~~ and equal dimensional

Proof :- Let T be injective.

Since $T(0) = \theta'$ in W, 0 is a pre-image of θ' and since T is injective, 0 is the only pre-image of θ'.

So, $\ker T = \{0\}$.

Conversely, let $\ker T = \{0\}$ and α, β be two elements of V such that $T(\alpha) = T(\beta)$ in W.

$$\therefore \theta' = T(\alpha) - T(\beta)$$
$$= T(\alpha - \beta), \text{ since } T \text{ is linear.}$$
$$\Rightarrow \alpha - \beta \in \ker T.$$

Since, $\ker T = \{0\}$
$\therefore \alpha = \beta$

Thus, $T(\alpha) = T(\beta) \Rightarrow \alpha = \beta$ and therefore T is injective.

Hence, the proof.

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Theorem:- Let V and W be two finite and equal dimensional vector space. $T: V \to W$ be a LT. Then the following are equivalent.

(i) T is one-one
(ii) T is onto
(iii) $Im(T) = W$.

Proof:- Case-I:- Let T is one to one.

Then $\ker T = \{0\}$

$\therefore \dim(\ker T) = 0$

$\therefore$ From Rank-dimension-theorem,

$\dim(Im(T)) = \dim V = \dim W$. $[\because \dim(\ker T) = 0$

$\therefore Im(T) = W$ $[\because R(T)$ is subspace of $W]$

and T is onto.

Case-II:- Let T is onto.

Then $Image(T) = W$.

From Sylvester's law, we have

$\dim(\ker T) + \dim(Image(T)) = \dim(V) = \dim W$

$\Rightarrow \dim(\ker T) + \dim(W) = \dim W$ $\left[\because Image(T) = W \text{ and } \dim V = \dim W\right]$

$\Rightarrow \dim(\ker T) = 0$

$\Rightarrow \ker T = \{\theta\}$.

Hence, T is one-to-one.

Case-III:- Let $Im(T) = W$.

Then it is obvious that T is onto.

From Sylvester's law, we have

$\dim(\ker T) + \dim(Im(T)) = \dim V = \dim W$ $[\because V$ and W are equal dimensional vector spaces$]$

$\Rightarrow \dim(\ker T) + \dim(W) = \dim(W)$ $[\because Im(T) = W]$

$\Rightarrow \dim(\ker T) = 0$

$\Rightarrow \ker T = \{\theta\}$.

Hence, T is one-to-one.

Ex:- Let $T: P_2(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ be a linear transformation defined by

$$T(f(x)) = 2f'(x) + \int_0^x 3f(t)\,dt.$$

$P_n(\mathbb{R}) = \{$ The set of all polynomials upto order n and coefficients from $\mathbb{R}$ $\}$

Soln:- Now, $\{1, x, x^2\}$ is the standard basis of $P_2(\mathbb{R})$.

Now, $T(1) = 3x$

$T(x) = 2 + \frac{3}{2}x^2$

$T(x^2) = 4x + x^3$.

$\therefore R(T) = \text{span}\{3x,\ 2+\frac{3}{2}x^2,\ 4x+x^3\}$ $\boxed{R(T) = \text{Image}(T)}$

and $\{3x,\ 2+\frac{3}{2}x^2,\ 4x+x^3\}$ is L.I.

$\therefore \{3x,\ 2+\frac{3}{2}x^2,\ 4x+x^3\}$ is the basis of $R(T)$.

$\therefore \dim(R(T)) = 3$.

From Rank dimension theorem,

$\dim(\ker(T)) + \dim(R(T)) = \dim P_2(\mathbb{R})$

$\Rightarrow \dim(\ker T) = 0$

$\therefore T$ is one-to-one.

Theorem:- Let V and W be two vector spaces with V be finite dimensional and $T_1, T_2 : V \longrightarrow W$ be two LT. Let $\{v_1, v_2, \dots, v_n\}$ be basis of V and $T_1(v_i) = T_2(v_i)$, $i = 1, 2, \dots, n$. Then $T_1 = T_2$.

Proof:- Let $x \in V$.

$\therefore x = c_1v_1 + c_2v_2 + \dots + c_nv_n$

$\therefore T_1(x) = T_1(c_1v_1 + c_2v_2 + \dots + c_nv_n)$

$= c_1T_1(v_1) + c_2T_1(v_2) + \dots + c_nT_1(v_n)$

$= c_1T_2(v_1) + c_2T_2(v_2) + \dots + c_nT_2(v_n)$

$= T_2(c_1v_1 + c_2v_2 + \dots + c_nv_n)$ $[\because T_1(v_i) = T_2(v_i),\ i = 1, 2, \dots]$

$= T_2(x)$.

Since, x is arbitrary element of V,

So, $T_1(x) = T_2(x)$, $\forall x \in V$.

Hence, $T_1 = T_2$.

Theorem :- Let $\{v_1, v_2, \cdots, v_n\}$ be a basis for V and $T: V \to W$ be one-to-one and onto linear mapping. Then the set $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is basis of W.

Proof :- We have, $\{v_1, v_2, \cdots, v_n\}$ is a basis of V.

Let $c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n) = 0$, $c_i \in F$, $i = 1(1)n$.

$\Rightarrow T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = 0$

$\Rightarrow c_1v_1 + c_2v_2 + \cdots + c_nv_n \in \ker(T)$ —— (1)

Since, T is one-to-one;

then $\ker(T) = \{0\}$.

$\therefore$ From (1),

$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$

~~Sin.~~ $\Rightarrow c_1 = c_2 = \cdots = c_n = 0$ [$\because$ $\{v_1, v_2, \cdots, v_n\}$ is a basis of V]

Thus, $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is ~~a basis of W~~ linearly independent.

Since, $\{v_1, v_2, \cdots, v_n\}$ is a basis of V

then $\text{span}\{T(v_1), T(v_2), \cdots, T(v_n)\} = \text{Image}(T)$

Since, T is onto then $\text{Image}(T) = W$

Thus, $\text{span}\{T(v_1), T(v_2), \cdots, T(v_n)\} = W$

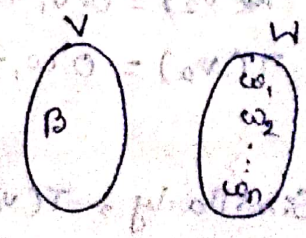
Hence, $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is a basis of W.

Hence, the proof.

Theorem :- Let V and W be two vector spaces and V be finite dimensional. If $\beta = \{v_1, v_2, \cdots, v_n\}$ be a unique basis of V and $\{\omega_1, \omega_2, \cdots, \omega_n\}$ are in W, then $\exists$ a linear transformation $T: V \to W$ such that $T(v_i) = \omega_i$, $i = 1, 2, \cdots, n$.

Proof :- Let $x \in V$.

Then $x = c_1v_1 + c_2v_2 + \cdots + c_nv_n$



Now, the expression of $x \in V$ is unique because

if $x = d_1v_1 + d_2v_2 + \cdots + d_nv_n$

$\therefore c_1v_1 + c_2v_2 + \cdots + c_nv_n = d_1v_1 + d_2v_2 + \cdots + d_nv_n$

$\Rightarrow (c_1 - d_1)v_1 + (c_2 - d_2)v_2 + \cdots + (c_n - d_n)v_n = 0$

Now, β is basis of V.

$\therefore c_1 - d_1 = 0, \; c_2 - d_2 = 0, \; \cdots, \; c_n - d_n = 0$

i.e., $c_i = d_i, \; i = 1, 2, \cdots, n$.

Thus, the expression $x = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ is unique.

Consider a mapping $T: V \to W$ by

$T(x) = c_1\omega_1 + c_2\omega_2 + \cdots + c_n\omega_n$

Now, we have to show T is linear.

Let $x, y \in V$ and $a \in F$, and $y = d_1v_1 + d_2v_2 + \cdots + d_nv_n$, $x \neq y$

$\therefore ax + y$
$= a(c_1v_1 + c_2v_2 + \cdots + c_nv_n) + (d_1v_1 + d_2v_2 + \cdots + d_nv_n)$
$= (ac_1 + d_1)v_1 + (ac_2 + d_2)v_2 + \cdots + (ac_n + d_n)v_n$

$\therefore T(ax + y)$
$= T\{(ac_1 + d_1)v_1 + (ac_2 + d_2)v_2 + \cdots + (ac_n + d_n)v_n\}$
$= \cancel{(ac_1 + d_1)T(v_1) + (ac_2 + d_2)T(v_2) + \cdots + (ac_n + d_n)T(v_n)}$
$= (ac_1 + d_1)\omega_1 + (ac_2 + d_2)\omega_2 + \cdots + (ac_n + d_n)\omega_n$
$= (ac_1\omega_1 + ac_2\omega_2 + \cdots + ac_n\omega_n) + (d_1\omega_1 + d_2\omega_2 + \cdots + d_n\omega_n)$
$= a\{c_1\omega_1 + c_2\omega_2 + \cdots + c_n\omega_n\} + \{d_1\omega_1 + d_2\omega_2 + \cdots + d_n\omega_n\}$
$= aT(x) + T(y)$ $\quad [\because T(y) = T(d_1v_1 + d_2v_2 + \cdots + d_nv_n) = d_1\omega_1 + d_2\omega_2 + \cdots + d_n\omega_n]$

$\therefore T$ is linear.

Now, $v_1, v_2, \cdots, v_n \in V$.

$v_1 = 1 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n$

$T(v_1) = 1 \cdot \omega_1 + 0 \cdot \omega_2 + \cdots + 0 \cdot \omega_n = \omega_1$

$v_2 = 0 \cdot v_1 + 1 \cdot v_2 + \cdots + 0 \cdot v_n$

$T(v_2) = 0 \cdot \omega_1 + 1 \cdot \omega_2 + \cdots + 0 \cdot \omega_n = \omega_2$

$\vdots$

Similarly, $T(v_n) = \omega_n$

$\therefore T(v_i) = \omega_i, \; i = 1, 2, \cdots, n$.

Let $U: V \to W$ is another LT s.t. $U(v_i) = \omega_i, i = 1, 2, \ldots, n$.

Now, $x \in V$,

Then $x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

$$\therefore U(x) = U(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$$
$$= c_1 U(v_1) + c_2 U(v_2) + \cdots + c_n U(v_n)$$
$$= c_1 \omega_1 + c_2 \omega_2 + \cdots + c_n \omega_n$$
$$= T(x)$$

Thus, $U(x) = T(x)$, for all $x \in V$.

This implies that $U = T$.

Hence, the proof.

$\boxed{\text{NOTE}}$:-

(i) W need not to be finite

(ii) $\omega_1, \omega_2, \ldots, \omega_n$ need not to be a basis of W.

(iii) $\omega_1, \omega_2, \ldots, \omega_n$ need ~~to~~ not to be different always.

⊛ A linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(x_1, x_2, x_3) = \left(\frac{x_1 + x_2 + x_3}{2}, \frac{x_1 + x_2 + x_3}{2}, \frac{x_1 + x_2 + x_3}{2} \right).$$

Examine whether ~~Show that~~ T is bijective or not.

Soln:- $\ker T = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : T(x_1, x_2, x_3) = 0 \}$

$\Rightarrow x_1 + x_2 + x_3 = 0$.

Let $x_2 = a, x_3 = b \in \mathbb{R}$.

$\therefore x_1 = -(a+b)$.

$$\therefore T(x_1, x_2, x_3) = (-a-b, a, b)$$
$$= a(-1,1,0) + b(-1,0,1), \; a, b \in \mathbb{R}.$$

$\therefore \ker T = \text{span}\{(-1,1,0), (-1,0,1)\}$

and $\{(-1,1,0), (-1,0,1)\}$ is L.I.

$\therefore \dim(\ker T) = 2$

$\therefore T$ is not one-to-one.

$\dim\{\text{Image}(T)\} = 3 - 2 = 1 \neq \dim(\mathbb{R}^3)$

$\therefore \text{Image}(T) \neq \mathbb{R}^3$

$\therefore T$ is not onto.

Hence, T is not bijective.

(*) Determine the linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vector $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ to the vectors $\{(2,0,0), (0,2,0), (0,0,2)\}$ resp. Find $\ker T$ and Image(T).

Soln. - Let $x \in \mathbb{R}^3$, $x = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$, $c_1, c_2, c_3 \in \mathbb{R}$.

$$\Rightarrow T(x) = c_1(2,0,0) + c_2(0,2,0) + c_3(0,0,2).$$
$$= (2c_1, 2c_2, 2c_3)$$

$$x = (c_2+c_3, c_1+c_3, c_1+c_2)$$
$$= (x_1, x_2, x_3)$$

$\therefore c_2 + c_3 = x_1, \quad c_1 + c_3 = x_2, \quad c_1 + c_2 = x_3$

$\therefore c_1 - c_2 = x_2 - x_1$

$c_1 + c_2 = x_3$

Adding, $c_1 = \dfrac{x_2+x_3-x_1}{2}$

$\therefore c_2 = \dfrac{x_1+x_3-x_2}{2}, \quad c_3 = \dfrac{x_1+x_2-x_3}{2}$

$\therefore T(x) = T(x_1, x_2, x_3)$

$$= (x_2+x_3-x_1, x_1+x_3-x_2, x_1+x_2-x_3).$$

$\ker T = \{x \in \mathbb{R}^3 : T(x) = \theta\}$

Now, $T(x) = 0$

$\Rightarrow x_2 + x_3 - x_1 = 0$

$x_1 + x_3 - x_2 = 0$

$x_1 + x_2 - x_3 = 0$

$\Rightarrow x_1 = x_2 = x_3 = 0.$

$\therefore \ker T = \{\theta\}$

$\therefore T$ is one-one.

$\dim V = \dim W$; for $V = \mathbb{R}^3$, $W = \mathbb{R}^3$

$\therefore T$ is onto.

$\Rightarrow$ Image(T) ~~$R(T)$~~ $= W = \mathbb{R}^3$.

(*) $T: \mathbb{R}^2 \to \mathbb{R}^3$ is linear s.t. $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$

Then find $T(8,11)$.

(*) Suppose that T is one-one and S is subset of V. Prove that S is L.I if and only if T(S) is L.I.

Proof :- Let $S = \{v_1, v_2, \ldots, v_n\}$

Let S is L.I.

Then $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$; $c_1, c_2, \ldots, c_n \in F$

$\Rightarrow c_1 = c_2 = \cdots = c_n = 0$

Let us consider $c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n) = 0$

$\Rightarrow T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = T(0)$ [$\because$ T is linear]

$\Rightarrow c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ [$\because$ T is one-one]

$\Rightarrow c_1 = c_2 = \cdots = c_n = 0$ [$\because$ S is L.I]

Thus, T(S) is L.I.

Conversely, let $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is L.I.

Let $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$

Then $c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n) = T(0) = 0$. [$\because$ T(S) is linear]

$\Rightarrow \cancel{T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = T(0)}$

$\Rightarrow c_1 = c_2 = \cdots = c_n = 0$ [$\because$ T(S) is linearly independent]

$\therefore$ S is L.I.

Problem :- $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(f(x)) = xf(x) + f'(x)$

Show that T is not bijective.

Matrix representation :-

Let $x \in V$ be any element and $\beta = \{v_1, v_2, \ldots, v_n\}$ basis of V.

Then $x \in V$, $x = a_1v_1 + a_2v_2 + \cdots + a_nv_n$.

Now the coordinate vector of $x \in V$ is denoted by $[x]_\beta$ and is defined as

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

(*) Let $V = P_2(\mathbb{R})$, $\{1, x, x^2\}$ is basis of $P_2(\mathbb{R})$ then $f(x) \in V$ and $f(x) = 1 + 2x + 4x^2$.

$\therefore$ The coordinate vector of $f(x)$ is $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

Definition:-

Let V and W be two finite dimensional vector spaces. Let $\beta = \{v_1, v_2, \dots, v_n\}$ is basis of V and $\gamma = \{\omega_1, \omega_2, \dots, \omega_m\}$ be a basis of W and $T: V \to W$ be a linear transformation. Then for each j, $1 \le j \le n$ $\exists$ uniquely scalars, $a_{ij} \in F$; $1 \le i \le m$ ~~$1 \le j \le n$~~ s.t.

$$T(v_j) = \sum_{i=1}^{m} a_{ij} \omega_i$$

$\therefore$ We get the matrix $(a_{ij})_{m \times n}$ and this is called matrix representation of T w.r.to the ordered basis β and γ.

This matrix is denoted is $[T]_\beta^\gamma$.

Now, if $V = W$ then $\beta = \gamma$. Then the matrix representation is denoted as $[T]_\beta$.

Ex:- Let $T: P_3(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$

$\therefore$ $\beta = \{1, x, x^2, x^3\}$ and $\gamma = \{1, x, x^2\}$ are the bases of $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ respectively and $T(f(x)) = f'(x)$.

Soln:-

$T(1) = 0 = 0.1 + 0.x + 0.x^2$

$T(x) = 1 = 1.1 + 0.x + 0.x^2$

$T(x^2) = 2x = 0.1 + 2.x + 0.x^2$

$T(x^3) = 3x^2 = 0.1 + 0.x + 3.x^2$

$$\therefore [T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Ex:- Let $T: P_3(\mathbb{R}) \to P_3(\mathbb{R})$ a. Let $\beta = \{1, x, x^2, x^3\}$ and $\gamma = \{1, x, x^2, x^3\}$ are the bases of $P_3(\mathbb{R})$ and $P_3(\mathbb{R})$ resp. and $T(f(x)) = f'(x) + f(x)$.

Solⁿ:- $T(1) = 0+1 = 1\cdot1 + x\cdot0 + 0\cdot x^2 + 0\cdot x^3$

$T(x) = 1 + x = 1\cdot1 + 1\cdot x + 0\cdot x^2 + 0\cdot x^3$

$T(x^2) = 2x + x^2 = 0\cdot1 + 2\cdot x + 1\cdot x^2 + 0\cdot x^3$

$T(x^3) = 3x^2 + x^3 = 0\cdot1 + 0\cdot x + 3\cdot x^2 + 1\cdot x^3$

$$\therefore [T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Ex:- $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$.

Find $[T]_\beta^\gamma$

Solⁿ:- $T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$

$\{(1,0), (0,1)\}$ are bases of $\mathbb{R}^2$ and $\{(1,0,0), (0,1,0), (0,0,1)\}$ are bases of $\mathbb{R}^3$.

$\therefore T(1,0) = (1, 0, 2)$.

$T(0,1) = (3, 0, -4)$

$$\therefore [T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}$$

(*) $T: \mathbb{R}^2 \to \mathbb{R}^3$ is linear such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. Then find $T(8,11)$.

↳ Let $\xi = (x, y)$ be an arbitrary element in $\mathbb{R}^2$.

$\therefore \xi = c_1(1,1) + c_2(2,3)$

$= (c_1 + 2c_2, c_1 + 3c_2)$

$\therefore x = c_1 + 2c_2$

$y = c_1 + 3c_2$

Solving, $c_1 = 3x - 2y$, $c_2 = y - x$.

$\therefore T(\xi) = c_1 T(1,1) + c_2 T(2,3)$ [$\because T$ is linear]

$= c_1(1,0,2) + c_2(1,-1,4)$

$= (c_1 + c_2, -c_2, 2c_1 + 4c_2)$

$= (2x-y, x-y, 2x)$

Hence, $T(6,11) = (5,-3,16)$

Problem:- $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(f(x)) = xf(x) + f'(x)$.
Show that T is not bijective.

Soln:- $T(f(x)) = xf(x) + f'(x)$

Let $f(x) \in \ker T$

$\therefore T(f(x)) = 0$

$\Rightarrow xf(x) + f'(x) = 0$

$\Rightarrow \dfrac{f'(x)}{f(x)} = -x$

int, $\log f(x) = -\dfrac{x^2}{2} + c$

$\Rightarrow f(x) = e^{(-\frac{x^2}{2}+c)}$

But $e^{(-\frac{x^2}{2}+c)}$ is not a polynomial.

Thus, $f(x) = 0$.

$\therefore \ker T = \{0\}$.

$\therefore T$ is one-one.

$\{1, x, x^2\}$ order basis of $P_2(\mathbb{R})$.

$T(1) = x$

$T(x) = x \cdot x + 1 = x^2 + 1$

$T(x^2) = x \cdot x^2 + 2x = x^3 + 2x$.

$\therefore \{x, x^2+1, x^3+2x\}$ spans $R(T)$

and $\{x, x^2+1, x^3+2x\}$ is L.I.

But ~~Im(T)~~ = dim(Image(T)) $= 3 \neq \dim P_3(\mathbb{R})$

$\therefore T$ is not onto.

Hence, T is not bijective.

Ex:- $T: \mathbb{R}^3 \to \mathbb{R}^2$ is linear defined by

$$T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3,\ x_1 - 3x_2 - 2x_3)$$

Find the matrix of T relative to ordered basis

(i) $\{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$ and $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$.

(ii) $\{(0,1,0), (1,0,0), (0,0,1)\}$ of $\mathbb{R}^3$ and $\{(0,1), (1,0)\}$ of $\mathbb{R}^2$.

(i) $A = \begin{pmatrix} 3 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$

(ii) $A = \begin{pmatrix} -3 & 3 & -2 \\ -2 & 1 & 1 \end{pmatrix}$

Sol^n:- $T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3,\ x_1 - 3x_2 - 2x_3)$

(i) $T(1,0,0) = (3,1) = 3(1,0) + 1(0,1)$

$T(0,1,0) = (-2,-3) = -2(1,0) - 3(0,1)$

$T(0,0,1) = (1,-2) = 1(1,0) - 2(0,1)$

$\therefore$ The required matrix is

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & -3 & -2 \end{bmatrix}$$

(ii)

$T(0,1,0) = (-2,-3) = \cancel{\{-2(1,0}} - 3(0,1) - 2(1,0)$

$T(1,0,0) = (3,1) = 1(0,1) + 3(1,0)$

$T(0,0,1) = (1,-2) = -2(0,1) + 1(1,0)$

$\therefore$ The required matrix is

$$A = \begin{bmatrix} -3 & 1 & -2 \\ -2 & 3 & 1 \end{bmatrix}$$

Ex: The matrix of a linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^2$ relative to the ordered basis $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,0), (1,1)\}$ of $\mathbb{R}^2$ is $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$. Find the linear transformation and matrix of T relative to the ordered basis $\{(1,1,0), (1,0,1), (0,1,1)\}$ of $\mathbb{R}^3$ and $\{(1,1), (0,1)\}$ of $\mathbb{R}^2$.

Ans:- $T(x_1, x_2, x_3) = (2x_1 + 2x_2 + x_3,\ \frac{1}{2}(-x_1 + x_2 + 3x_3))$; $A = \begin{pmatrix} 4 & 3 & 3 \\ -4 & -2 & -1 \end{pmatrix}$

Sol^n.-

$T(0,1,1) = 1(1,0) + 2(0,1) = (3,2)$

$T(1,0,1) = 2(1,0) + 1(0,1) = (3,1)$

$T(1,1,0) = 4(1,0) + 0(0,1) = (4,0)$

Let $\xi = (x,y,z)$ be an arbitrary element in $\mathbb{R}^3$.

$\therefore (x,y,z) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$, $c_i \in \mathbb{R}$, $i = 1(1)3$

$= (c_2 + c_3,\ c_1 + c_3,\ c_1 + c_2)$

$\therefore c_2 + c_3 = x \qquad c_1 + c_3 = y \qquad c_1 + c_2 = z$

$c_2 + c_3 - c_1 - c_3 = x - y$

$\Rightarrow c_2 - c_1 = x - y$

$c_2 + c_1 = z$

$\Rightarrow c_2 = \frac{1}{2}(x - y + z)$

$\therefore c_1 = z - \frac{1}{2}(x - y + z) = -\frac{1}{2}(x - y - z)$

$\therefore c_3 = x - c_2$

$= x - \frac{1}{2}(x - y + z) = \frac{1}{2}(x + y - z)$

$\therefore (x,y,z) = \cancel{\frac{1}{2}(x+y-z)(0,1,1)}$
$-\frac{1}{2}(x - y - z)(0,1,1) + \frac{1}{2}(x - y + z)(1,0,1) + \frac{1}{2}(x + y - z)(1,1,0)$

$T(x,y,z) = -\frac{1}{2}(x - y - z)T(0,1,1) + \frac{1}{2}(x - y + z)T(1,0,1) + \frac{1}{2}(x + y - z)T(1,1,0)$

$= -\frac{1}{2}(x - y - z)(3,2) + \frac{1}{2}(x - y + z)(3,1) + \frac{1}{2}(x + y - z)(4,0)$

$= \left(\frac{5}{2}x + \frac{3}{2}y - \frac{3}{2}z,\ -\frac{x}{2} + \frac{1}{2}y - \frac{1}{2}z\right)$

$= \left(\frac{1}{2}(5x + 3y - 3z),\ \cancel{-\frac{1}{2}(x - y + z)}\right)$

$= \left(2x + 2y + z,\ \frac{1}{2}(-x + y + 3z)\right)$

$T(1,1,0) = (4,0) = c_1(1,1) + c_2(0,1) = 4(1,1) - 4(0,1)$

$T(1,0,1) = (3,1) = c_3(1,1) + c_4(0,1) = 3(1,1) - 2(0,1)$

$T(0,1,1) = \cancel{(0,0)}\ (3,2) = \cancel{0(1,1)}\ c_5(1,1) + c_6(0,1) = 3(1,1) - 1(0,1)$

$\therefore$ The required matrix is

$$A = \begin{bmatrix} 4 & 3 & 3 \\ -4 & -2 & -1 \end{bmatrix}$$

Definition :- Let V and W be two vector spaces and T, U are two linear transformation from V to W; i.e, $T, U: V \to W$

Then the addition of two linear transformation

$$(T+U): V \to W$$ is also a linear transformation from V to W and defined as

$$(T+U)(x) = T(x) + U(x)$$

and $(aT)(x) = a\,T(x)$, $\forall\, x \in V$

Theorem :- Let V and W be two vector spaces over a field F. $T, U: V \to W$ be linear, then show that $aT + U$ is linear, for $a \in F$.

Proof :- Let $x, y \in V$

Then $cx + y \in V$; $c \in F$

$$(aT+U)(cx+y) = (aT)(cx+y) + U(cx+y) \quad \text{[addition of two linear transform]}$$
$$= a\,T(cx+y) + U(cx+y)$$
$$= a[c\,T(x) + T(y)] + [c\,U(x) + U(y)]$$
$$= c[aT(x) + U(x)] + [aT(y) + U(y)]$$
$$= c\,(aT+U)(x) + (aT+U)(y)$$

Problem :- $\dim V = n$, $\dim W = m$, $T: V \to W$ be linear

(i) If $n > m$, then T is not injective.

(ii) If $m > n$, then T is not surjective.

$R(T) = \mathrm{Im}(T)$

↳ Let T is injective.

Then $\ker T = \{0\}$

i.e, $\dim \ker T = 0$

$\Rightarrow \text{Nullity}(T) = 0$

and $\text{Nullity}(T) + \text{Rank}(T) = \dim V$

$\Rightarrow \text{Rank}(T) = \dim V = n$ ——— (1)

Now, $\text{Image}(T) \subseteq W$

$\Rightarrow \dim(\text{Image}(T)) \le \dim W = m$

$\Rightarrow \text{Rank}(T) \le m$

So, $n \le m$

$\therefore T$ is not injective for $n > m$

(ii) Let T is surjective

Then $\dim(Im(T)) = \dim W = m$

$\therefore \dim(\ker T) + \cancel{\dim} \; m = \dim V = n$

$\Rightarrow \dim(\ker T) = n-m$

Now, $\dim(\ker T) \geq 0$

$\Rightarrow n-m \geq 0$

$\Rightarrow n \geq m$.

$\therefore$ T is not surjective for $m > n$.

<u>Problem</u>: Let $T: V \to V$ s.t. $T^2 = 0$. What can say about the relation $R(T)$ to the null space $N(T)$? $\boxed{R(T) \Rightarrow Im(T}$

Give an example of a linear operator T of $\mathbb{R}^2$ s.t. $T^2 = 0$ but $T \neq 0$.

<u>Sol^n</u>:- ~~Now~~ $v \in V$, $T^2(v) = 0$.

$\Rightarrow T(T(v)) = 0$

or, $T(v) \in N(T)$

but, $T(v) \in R(T)$

$\therefore R(T) \subseteq N(T)$

<u>Example</u>:- $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x_1, x_2) = (x_2, 0)$.

$$T^2(x_1, x_2) = T(T(x_1, x_2))$$
$$= T(x_2, 0)$$
$$= (0, 0)$$

$\therefore T^2 = 0$.

(*) A linear transformation $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}.$$

Compute the matrix $[T]_\beta^\gamma$, where $\beta = \{1, x, x^2\}$ and $\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

(*) Let $T: V \to V$ be a linear operator and $\text{Rank } T^2 = \text{Rank } T$. Then show that $\text{Range } T \cap \ker T = \{0\}$ i.e., $\{R(T) \cap N(T)\} = \{0\}$.

<u>Proof</u>:- From rank-dimension theorem,

$\dim \ker T + \dim \text{Range } T = \dim V$

~~and~~ $\dim \ker T^2 + \dim \text{Range } T^2 = \dim V$ [$\because T^2$ is also a linear o

$\therefore \dim \ker T = \dim \ker T^2$ ——(1) $[\because \text{Rank } T^2 = \text{Rank } T]$

Now, we have to show that $\ker T = \ker T^2$.

$\therefore x \in \ker T$

$\therefore T(x) = 0$.

$\therefore T^2(x) = T(T(x)) = T(0) = 0$.

$\therefore T^2(x) = 0$.

$\Rightarrow x \in \ker T^2$.

$\therefore \ker T \subseteq \ker T^2$

From (1), $\ker T = \ker T^2$ ——(2)

Let $x \in R(T) \cap N(T)$

$\therefore x \in R(T)$ and $x \in N(T) = N(T^2)$

$\cancel{\therefore x \in N(T) \Rightarrow T(x) = 0}$.

and $x \in R(T) \Rightarrow x = T(y)$, for $y \in V$.

$\cancel{\Rightarrow T(x) = T^2(y)}$

$\cancel{T(x) = 0}$

$\cancel{\Rightarrow T^2(x) = 0}$.

$\cancel{\Rightarrow T(T(x)) = 0}$

Now, $x \in N(T)$ $\cancel{\Rightarrow x \in N(T)}$

$\Rightarrow T(x) = 0$.

$T^2(y) = T(x) = 0$.

$\Rightarrow T^2(y) = 0$

$\Rightarrow y \in N(T^2) = N(T)$

$\Rightarrow T(y) = 0$

$\Rightarrow x = 0$.

~~Hence~~, Since, ~~x is~~ an arbitrary point of $R(T) \cap N(T)$.

Then $R(T) \cap N(T) = \{0\}$

Theorem:- Let V and W be two finite dimensional vector spaces with ordered bases β and γ respectively. Let ~~S,~~ $S, T : V \rightarrow W$ be two LT. Then

(i) $[S+T]_\beta^\gamma = [S]_\beta^\gamma + [T]_\beta^\gamma$

(ii) $[aT]_\beta^\gamma = a[T]_\beta^\gamma$, $a \in F$.

Proof: Let $\beta = \{v_1, v_2, \ldots, v_n\}$, $\gamma = \{\omega_1, \omega_2, \ldots, \omega_m\}$.

Now, $T(v_j) = \sum_{i=1}^{m} a_{ij}\,\omega_i \Rightarrow [T]_\beta^\gamma = (a_{ij})$, $j = 1, 2, \ldots, n$

and $S(v_j) = \sum_{i=1}^{m} b_{ij}\,\omega_i$, $j = 1, 2, \ldots, n$.

$\Rightarrow [S]_\beta^\gamma = (b_{ij})$

, where a_{ij} and b_{ij} are unique scalars.

(i) :- ~~$[S+T]$~~ $(S+T)(v_j) = S(v_j) + T(v_j)$ [Addition of two LT]

$$= \sum_{i=1}^{m} b_{ij}\,\omega_i + \sum_{i=1}^{m} a_{ij}\,\omega_i$$

$$= \sum_{i=1}^{m} (b_{ij} + a_{ij})\,\omega_i$$

$\therefore [S+T]_\beta^\gamma = b_{ij} + a_{ij}$

$$= [S]_\beta^\gamma + [T]_\beta^\gamma$$

(ii) $(aT)(v_j) = aT(v_j)$

$$= a \sum_{i=1}^{m} a_{ij}\,\omega_i$$

$\therefore [aT]_\beta^\gamma = a\,(a_{ij})$

$$= a[T]_\beta^\gamma \quad , a \in F$$

Example: Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and $S : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be two LT defined by

$T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$ and

$S(a_1, a_2) = (a_1 - a_2, 2a_1, 3a_1 + 2a_2)$

$\therefore (S+T)(a_1, a_2) = S(a_1, a_2) + T(a_1, a_2)$

$$= (2a_1 + 2a_2, 2a_1, 5a_1 - 2a_2)$$

$\beta = \{(1,0),(0,1)\}$, $\gamma = \{(1,0,0),(0,1,0),(0,0,1)\}$

$$\therefore [S+T]_\beta^\gamma = \begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{pmatrix}$$

Now, $[S]_\beta^\gamma = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{pmatrix}$ and $[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}$

$$\therefore [S]_\beta^\gamma + [T]_\beta^\gamma = \begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{pmatrix} = [S+T]_\beta^\gamma$$

Definition :- Let S, T are two L.T from V to W. i.e, $S, T: V \to W$. Then multiplication of two ~~LT~~ LT is defined as
$(ST)(x) = S(T(x))$, $\forall x \in V$.

Theorem :- Let $T: V \to W$ and $S: W \to Z$ be LT. Let β, γ, δ are the ordered bases of V, W and Z respectively. Then $[ST]_\beta^\delta = [S]_\gamma^\delta \, [T]_\beta^\gamma$.

Proof :- Let, $\beta = \{v_1, v_2, \dots, v_n\}$, $\gamma = \{w_1, w_2, \dots, w_m\}$ and $\delta = \{z_1, z_2, \dots, z_p\}$ be the ordered bases.

Let, ~~A=~~ $T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$, $A = [T]_\beta^\gamma = (a_{ij})_{m\times n}$ and ; $j = 1, 2, \dots, n$

~~$T(w_j)$~~ $S(w_j) = \sum_{i=1}^{p} b_{ij} z_i$, $j = 1, 2, \dots, m$.

and $B = [S]_\gamma^\delta = (b_{ij})_{p\times m}$

Now, $(ST)(v_k) = S(a_{1k}w_1 + a_{2k}w_2 + \dots + a_{mk}w_m)$, $k = 1,2 \dots n$

$= a_{1k}S(w_1) + a_{2k}S(w_2) + \dots + a_{mk}S(w_m)$

$= a_{1k}(b_{11}z_1 + b_{21}z_2 + \dots + b_{p1}z_p)$
$+ a_{2k}(b_{12}z_1 + b_{22}z_2 + \dots + b_{p2}z_p)$
$+ \dots \dots$
$+ a_{mk}(b_{1m}z_1 + b_{2m}z_2 + \dots + b_{pm}z_p)$

$= (b_{11}a_{1k} + b_{12}a_{2k} + \dots + b_{1m}a_{mk})z_1 + (b_{21}a_{1k} + b_{22}a_{2k} + \dots + b_{2m}a_{mk})z_2$
$+ \dots \dots + (b_{p1}a_{1k} + b_{p2}a_{2k} + \dots + b_{pm}a_{mk})z_p$

$= \sum_{i=1}^{p}\left(\sum_{j=1}^{m} b_{ij}a_{jk}\right) z_i$, $k = 1, 2, \dots, n$

$= \sum_{i=1}^{p} c_{ik} z_i$, $k = 1, 2, \dots, n$, where $c_{ik} = \sum_{j=1}^{m} b_{ij}a_{jk}$.

$$(ST)(v_k) = \sum_{i=1}^{p} c_{ik} z_i \quad , k = 1,2,\ldots n.$$

$$\Rightarrow [ST]_\beta^\delta = (c_{ik})_{p\times n}$$

The (i,k)-th element of $[S]_\gamma^\delta [T]_\beta^\gamma = \sum_{j=1}^{m} (b_{ij} a_{jk})$ $\quad {}_{, k=1,2,\ldots n}^{, i = 1,2,\ldots}$

$= c_{ik}$

$= (i,k)$-the element of $[ST]_\beta^\delta$ ⊛

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pm} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{m} b_{1j} a_{j1} & \sum_{j=1}^{m} b_{1j} a_{j2} & \cdots & \sum_{j=1}^{m} b_{1j} a_{jn} \\ \vdots & & \sum_{j=1}^{m} b_{ij} a_{jk} = c_{ik} & \\ \sum_{j=1}^{m} b_{pj} a_{j1} & \sum_{j=1}^{m} b_{pj} a_{j2} & \cdots & \sum_{j=1}^{m} b_{pj} a_{jn} \end{pmatrix}$$

Ex:- $S: P_3(R) \to P_2(R)$ and $T: P_2(R) \to P_3(R)$ be two LT defined by $S(f(x)) = f'(x)$, $T(f(x)) = \int_0^x f(t)\,dt$

Solⁿ:- $(ST)f(x) = S(T(f(x))) = S\left(\int_0^x f(t)\,dt\right) = f(x)$

$\therefore ST = I$.

If $\beta = \{1, x, x^2, x^3\}$, $\delta = \{1, x, x^2\}$

$$\therefore [ST]_\beta^\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3}$$

($[ST]_\delta$)

Now, $[S]_\beta^\delta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

$S(1) = 0$
$S(x) = 1$
$S(x^2) = 2x$
$S(x^3) = 3x^2$

$T(1) = x = 0\cdot 1 + 1\cdot x + 0\cdot x^2$
$T(x) = \frac{x^2}{2} = 0\cdot 1 + 0\cdot x + \frac{1}{2}x^2$
$T(x^2) = \frac{x^3}{3} = 0\cdot 1 + 0\cdot x + \frac{1}{3}x^3$

$$[T]_{S}^{\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$

Now $[S]_{\beta}^{S}[T]_{S}^{\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\therefore [ST]_{S} = [S]_{\beta}^{S}[T]_{S}^{\beta}$

⊛ A linear transformation $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by
$T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & 2f''(3) \end{pmatrix}$.
Compute the matrix $[T]_{\beta}^{\gamma}$, where $\beta = \{1, x, x^2\}$ and
$\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right.$.

Solⁿ:- $T(1) = \begin{pmatrix} 0 & 2\cdot 1 \\ 0 & 0 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Hence, $[T]_{\beta}^{\gamma} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Definition:- Let $V(F)$ and $W(F)$ be two vector spaces. Then $L(V, W)$ be the set of all linear transformation from V to W. If $V = W$ then it is denoted as $L(V)$. $L(V, W)$ is also a vector space over some field F.

⊛ Show that $L(V, W) \cong M_{m\times n}(F)$, where ~~we have~~ $\dim V = n$ and $\dim W = m$. Then show that $\dim(L(V, W)) = mn$.

Solⁿ:- Consider a mapping $\theta: L(V, W) \to M_{m\times n}(F)$ by
$\theta(T) = [T]_{\alpha}^{\beta}$, where $\alpha = \{v_1, v_2, \dots, v_n\}$ and $\beta = \{w_1, w_2, \dots, w_m\}$ are the ordered bases of V and W respectively.

Then θ is well-defined.

To show, θ is linear.

Let $a \in F$, $T_1, T_2 \in L(V, W)$ then

$$\theta(aT_1 + T_2) = [aT_1 + T_2]_\alpha^\beta$$
$$= a[T_1]_\alpha^\beta + [T_2]_\alpha^\beta$$
$$= a\,\theta(T_1) + \theta(T_2)$$

$\therefore$ θ is linear.

To show θ is one-one,

Let $\theta(S) = \theta(T)$, where $[S]_\alpha^\beta = (b_{ij})_{m\times n}$

$\Rightarrow [S]_\alpha^\beta = [T]_\alpha^\beta$ and $[T]_\alpha^\beta = (a_{ij})_{m\times n}$.

$\Rightarrow (b_{ij})_{m\times n} = (a_{ij})_{m\times n}$

$\Rightarrow b_{ij} = a_{ij}$, $\forall\, i, j$

Now, $S(v_j) = \sum_{i=1}^{m} b_{ij} w_i = \sum_{i=1}^{m} a_{ij} w_i$

$= T(v_j)$, $\forall\, j$

$\therefore S(v_j) = T(v_j)$, $\forall\, j$

$\Rightarrow S = T$.

$\therefore$ θ is one-one.

We have, $\theta : L(V,W) \to M_{m\times n}(F)$

For every $[T]_\alpha^\beta = (a_{ij})_{m\times n} \in M_{m\times n}(F)$, $\exists$ a linear transformation $T \in L(V,W)$ s.t.

$$\theta(T) = [T]_\alpha^\beta$$

$\therefore$ θ is onto.

Hence, $L(V,W) \cong M_{m\times n}(F)$

$\therefore \dim(L(V,W)) = mn$

$\boxed{\text{NOTE}}$:- If $V = W$,

then $\dim(L(V)) = m^2$ $[\because m = n]$

<u>Theorem</u> :- Let $V(F)$ and $W(F)$ be two finite dimensional vector spaces and $T: V \to W$ be a linear transformation. Then Rank of T = Rank of the matrix of T.

<u>Proof</u> :- Let $\alpha = \{v_1, v_2, \ldots, v_n\}$ and $\beta = \{w_1, w_2, \ldots, w_m\}$ be two ordered bases of V and W respectively.

Then, let $[T]_\alpha^\beta = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{pmatrix}$

Then $T(v_j) = \sum_{i=1}^{m} c_{ij} w_i$, $j = 1, 2, \ldots, n$.

$= c_{1j} w_1 + c_{2j} w_2 + \cdots + c_{mj} w_m$

Let Rank of $T = r$, i.e., ~~$\dim(\mathrm{Im}\,T) = r$~~ $\dim(R(T)) = r$, $r \leq m$

Now, $\{v_1, v_2, \ldots, v_n\}$ basis of V.

Then $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ generates $R(T)$.

i.e., $R(T) = \mathrm{Span}\{T(v_1), T(v_2), \ldots, T(v_n)\}$.

Without loss of generality, let us consider that $\{T(v_1), T(v_2), \ldots, T(v_r)\}$ is L.I set.

Now, consider an isomorphism ~~by~~ $\phi : W \to F^m$

by $\phi(w) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$

Now $\phi(T(v_1)) = \phi(c_{11} w_1 + c_{21} w_2 + \cdots + c_{m1} w_n)$

$= \begin{pmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{pmatrix}$

$\phi(T(v_2)) = \begin{pmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{m2} \end{pmatrix}, \ldots, \phi(T(v_r)) = \begin{pmatrix} c_{1r} \\ c_{2r} \\ \vdots \\ c_{mr} \end{pmatrix}$

Since, φ is an isomorphism (then φ is one-one and onto) and $\{T(v_1), T(v_2), \dots, T(v_r)\}$ is L.I set, then the set $\{\varphi(T(v_1)), \varphi(T(v_2)), \dots, \varphi(T(v_r))\}$ is also L.I.

Then the remaining elements $T(v_{r+1}), \dots, T(v_n)$ belong to the set of ~~span~~ $\cancel{\{T(v_1), T(v_2), \dots, T(v_n)\}}$

span $\{T(v_1), T(v_2), \dots, T(v_r)\}$.

Again since φ is an isomorphism then $\varphi(T(v_{r+1})), \dots \varphi(T(v_n))$ belong to the set of span $\{\varphi(T(v_1)), \varphi(T(v_2)), \dots, \varphi(T(v_r))\}$.

$$[T]_\alpha^\beta = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{pmatrix}$$

Hence, we see that first r number of ~~cto~~ column of the matrix $[T]_\alpha^\beta$ is L.I and remaining $(n-r)$ ~~cot~~ number of columns are belong to the span of first r number of columns of the matrix $[T]_\alpha^\beta$

Hence, Rank of the matrix $T = r$.

Invertibility and Isomorphism :-

Let V and W be two vector spaces and $T: V \to W$ be linear. A function $U: W \to V$ is said to inverse of T if

$$TU = I_W \text{ and } UT = I_V$$

If T has an inverse then T is called invertible.

If T is invertible then it is denoted by T^{-1} and it is unique and linear also.

Theorem :- Let V and W be two finite dimensional v.s. Then V be isomorphic to W iff $\dim V = \dim W$.

Non-singular :- Let V and W be the vector space over a field F. A linear mapping $T: V \to W$ is said to be non-singular if T is invertible.

Theorem:- Let V and W be two vector spaces and $T: V \to W$ be linear. Then T is invertible if and only if T is bijective.

Proof:- Let $T: V \to W$ be invertible.
Then $\exists$ a mapping $U: W \to V$ such that $UT = I_V$, $TU = I_W$.
To prove T is one-to-one, let $\alpha, \beta \in V$.

$$T(\alpha) = T(\beta) \Rightarrow UT(\alpha) = UT(\beta) \Rightarrow I_V(\alpha) = I_V(\beta) \Rightarrow \alpha = \beta$$

$\therefore$ T is one-to-one.
To prove T is onto, let $\gamma \in W$.
Since $TU = I_W$, we have $TU(\gamma) = \gamma$

i.e., $T\{U(\gamma)\} = \gamma$

This implies that $U(\gamma)$ is a pre-image of γ under T.
So, T is onto.
$\therefore$ T is bijective.
Conversely, let $T: V \to W$ be bijective.
Let $\alpha \in V$ and $T(\alpha) = \gamma$.
Since T is one-to-one, α is the unique pre-image of γ under T.
Since T is onto, each γ in W has a pre-image in V.
Let us define a mapping $U: W \to V$ by

$U(\gamma) = \alpha$ (the pre-image of γ under T), $\gamma \in W$.

Then $UT(\alpha) = U(\gamma) = \alpha$, $\forall \alpha \in V$
and $TU(\gamma) = T(\alpha) = \gamma$, $\forall \gamma \in W$.

$\therefore$ $UT = I_V$ and $TU = I_W$
and this proves that T is invertible.

Hence, the proof.

Theorem:- If $T: V \to W$ be linear and invertible, then T^{-1} ~~$W \to V$~~ is also linear.

Proof:- Let $\alpha', \beta' \in W$ and $T^{-1}(\alpha') = \alpha$, $T^{-1}(\beta') = \beta$.
Then $\alpha, \beta \in V$ and $T(\alpha) = \alpha'$, $T(\beta) = \beta'$.
Since T is linear, $T(\alpha + \beta) = T(\alpha) + T(\beta)$

$$= \alpha' + \beta'$$

$$\Rightarrow T^{-1}(\alpha' + \beta') = \alpha + \beta$$

$$= T^{-1}(\alpha') + T^{-1}(\beta')$$

Let $c \in F$
$\therefore$ $T(c\alpha) = cT(\alpha)$ [$\because$ T is linear]
$= c\alpha'$

$\Rightarrow T^{-1}(c\alpha') = c\alpha = cT^{-1}(\alpha')$, for all $c \in F$.

Hence, $T^{-1}: W \to V$ is linear.

Theorem:- Let V and W be two finite dimensional vector spaces over a field F. Then V is isomorphic to W if and only if $\dim V = \dim W$.

Proof:- Let V and W be isomorphic.

Then $\exists$ a linear mapping $T: V \to W$ such that T is both one-to-one and onto.

Since T is one-to-one, So $\ker T = \{\theta\}$.

and since T is onto, so $Im(T) = W$.

$\therefore \dim \ker T = 0$ and $\dim Im T = \dim W$.

Also, we know that

$$\dim \ker T + \dim Im T = \dim V$$

$$\text{Or,} \quad 0 + \dim W = \dim V$$

$$\text{Or,} \quad \dim W = \dim V$$

Conversely, let $\dim V = \dim W$.

Let $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a basis of V and $\{\beta_1, \beta_2, \cdots, \beta_n\}$ be a basis of W.

Then we know that $\exists$ a unique linear mapping $T: V \to W$ such that $T(\alpha_1) = \beta_1$, $T(\alpha_2) = \beta_2$, $\cdots$, $T(\alpha_n) = \beta_n$.

Let $\alpha \in \ker T$.

and $\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$, $c_i \in F$.

Then, $T(\alpha) = T(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n)$

$= c_1 T(\alpha_1) + c_2 T(\alpha_2) + \cdots + c_n T(\alpha_n)$ [$\because$ T is linear]

$= c_1\beta_1 + c_2\beta_2 + \cdots + c_n\beta_n$

Since, $T(\alpha) = \theta'$

So, $c_1\beta_1 + c_2\beta_2 + \cdots + c_n\beta_n = \theta'$

Since, $\{\beta_1, \beta_2, \cdots, \beta_n\}$ be the basis of W this implies that $c_1 = 0, c_2 = 0, \cdots, c_n = 0$.

$\therefore \alpha = \theta$

So, $\ker T = \{\theta\}$

Hence, T is one-to-one.

Also, we know that $\dim \ker T + \dim Im T = \dim V$

Or, $\dim Im T = \dim V$ [$\because \dim \ker T = 0$]

Or, $\dim Im T = \dim W$

$\therefore Im T = W$ [$\because Im(T) \subseteq W$]

Thus, T is onto.

$\therefore$ T is bijective.

Hence, V and W are isomorphic.

Theorem :- Let $T: V \to W$ be a L.T, V and W be two finite dimensional vector spaces with same dimension. Then the following are equivalent.

(i) T is invertible

(ii) T is non-singular

(iii) T is onto

(iv) If $\{v_1, v_2, \ldots, v_n\}$ be basis of V, then $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ basis of W.

Proof :- (i) $\to$ (ii)

T is invertible then T is non-singular

(ii) $\to$ (iii)

T is non-singular, implies T is onto.

(iii) $\to$ (i)

T is onto, dim V = dim W.

From Rank-nullity theorem,

$$\dim N(T) + \dim R(T) = \dim V$$

$$\Rightarrow \dim N(T) + \dim W = \dim W \quad \left[\because \dim R(T) = \dim W \text{ and } \dim V = \dim W\right]$$

$$\Rightarrow \dim N(T) = 0$$

$\therefore$ T is one-to-one

$\therefore$ T is invertible.

(i) $\to$ (iv), (i) $\to$ (iii) automatically)

T is invertible implies T is one-one and onto.

Then if $\{v_1, v_2, \ldots, v_n\}$ be basis of V therefore $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ will be basis of W.

(iii) $\to$ (i)

Let T is onto

$R(T) = W$

$\therefore$ From Rank-dimension theorem,

$$\dim R(T) + \dim N(T) = \dim V$$

$$\Rightarrow \dim W + \dim N(T) = \dim W \quad \left[\because \dim V = \dim W \text{ given}\right]$$

$$\Rightarrow \dim N(T) = 0$$

$$\to N(T) = \{0\}$$

$\to$ T is one-to-one.

Thus, T is invertible.

(iii) → (ii), let T is onto
⇒ T is invertible.
⇒ T is non-singular.

(iii) → (v), T is onto ⇒ $R(T) = W$ and $\dim V = \dim W$
Then from Rank-dimension-theorem,
$N(T) = \{0\}$
⇒ T is one-to-one
∴ T is bijective.
Since, $\{v_1, v_2, \ldots, v_n\}$ basis of V and T is bijective then $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is basis of W.

(iv) → (i)
Let $\{v_1, v_2, \ldots, v_n\}$ basis of V then $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ basis of W
Let $\alpha \in N(T)$.
$\therefore \alpha = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$
$\therefore T(\alpha) = T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$
$= c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$
Since, $\alpha \in N(T)$ then $T(\alpha) = \theta'$.
$\therefore c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n) = \theta'$
$\Rightarrow c_1 = c_2 = \cdots = c_n = 0$ [$\because \{T(v_1), T(v_2), \ldots, T(v_n)\}$ basis of W]
$\therefore \alpha = \theta$.
i.e. $N(T) = \{0\}$.
⇒ T is one-to one.
From Rank-dimension-theorem,
$\dim N(T) + \dim R(T) = \dim V$
$\Rightarrow 0 + \dim R(T) = \dim W$ [$\because \dim V = \dim W$]
$\Rightarrow R(T) = W$ [$\because R(T) \subseteq W$]
⇒ T is onto.
∴ T is bijective and hence T is invertible.

(iv) → (ii) Since, T is invertible, then T is non-singular.

Theorem :- Let $T : V \to W$ and $S : W \to U$ be two L.T. Then

(i) S and T 1-1 and onto then ST is 1-1 and onto.
Also $(ST)^{-1} = T^{-1} S^{-1}$.

(ii) If ST is 1-1, then T is 1-1.

(iii) If ST is onto, then S is also onto.

Proof :- (i) $ST(x) = ST(y)$, $x, y \in V$.
$\Rightarrow S(T(x)) = S(T(y))$
$\Rightarrow T(x) = T(y)$ [$\because$ S is 1-1]
$\Rightarrow x = y$ [$\because$ T is 1-1]
∴ ST is 1-1.

Now, $ST : V \to U$.
Let $u \in U$ be any element, $\exists$ $w \in W$ s.t.
$S(w) = u$ [$\because$ S is onto].

$\therefore \exists$ $v \in V$ s.t. $T(v) = w$ [$\because$ T is onto].

$\therefore ST(v) = S(\omega) = u$

$\therefore$ For any $u \in U$, $\exists$ $v \in V$ s.t. $ST(v) = u$.

$\therefore$ ST is onto.

$\therefore$ ST is invertible, i.e., $(ST)^{-1}$ exists.

Now, $(ST)(T^{-1}S^{-1}) = S(TT^{-1})S^{-1} = SS^{-1} = I$.

Again, $(T^{-1}S^{-1})(ST) = T^{-1}(S^{-1}S)T = T^{-1}T = I$

$\therefore (ST)^{-1} = T^{-1}S^{-1}$

(ii) Let $v \in \ker T$

Then $T(v) = 0$.

$\therefore ST(v) = S(0) = 0$

$\Rightarrow v \in \ker ST$, but ST is 1-1

$\therefore v = 0$.

$\therefore \ker T = \{0\}$.

Thus, T is 1-1.

(iii) $ST : V \to U$.

By definition, for $u \in U$, $\exists$ $v \in V$ s.t.

$ST(v) = u$.

$\Rightarrow S(T(v)) = u$.

Now, $T(v) \in W$.

let $T(v) = \omega$, where $\omega \in W$.

$\therefore S(\omega) = u$.

$\therefore$ For any $u \in U$, $\exists$ $\omega \in W$ s.t. $S(\omega) = u$.

$\therefore$ S is onto.

Theorem :- Let $T : V(F) \to V(F)$ be a L.T. Let $\beta = \{u_1, u_2, \ldots, u_n\}$ and $\beta' = \{v_1, v_2, \ldots, v_n\}$ be two ordered basis of V then there exist a non-singular matrix P s.t. $[T]_{\beta'} = P^{-1}[T]_{\beta}P$.

Proof :- Let, $S : V(F) \to V(F)$ be a L.T s.t.

$S(u_i) = v_i$, $\forall i = 1, 2, \ldots, n$.

Let $x \in \ker S$

Then $S(x) = 0$.

Now $x \in V(F)$, $x = c_1u_1 + c_2u_2 + \cdots + c_nu_n$, $c_i \in F$, $i = 1, 2, \ldots, n$.

$\therefore S(x) = c_1S(u_1) + c_2S(u_2) + \cdots + c_nS(u_n) = 0$.

Or, $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$.

But $\beta' = \{v_1, v_2, \ldots, v_n\}$ be basis of $V(F)$,

then $c_i = 0$, $\forall\, i = 1(1)n$.

$\therefore x = 0$.

Thus, $\ker S = \{0\}$

$\Rightarrow$ S is 1-1 and dimension are equal,

then S is also onto.

$\therefore$ S is invertible.

Let $[T]_\beta = (a_{ij})$

and $T(u_j) = \sum_{i=1}^{n} a_{ij} u_i$, $j = 1, 2, \ldots, n$.

$$(STS^{-1})(v_j) = ST(S^{-1}(v_j)) = ST(u_j) \quad [\because S(u_j) = v_j]$$

$$= S\left(\sum_{i=1}^{n} a_{ij} u_i\right)$$

$$= \sum_{i=1}^{n} a_{ij}\, S(u_i)$$

$$= \sum_{i=1}^{n} a_{ij}\, v_i \quad [\because S(u_i) = v_i]$$

chec

$\therefore [STS^{-1}]_{\beta'} = (a_{ij}) = [T]_\beta$.

$[S]_{\beta'} [T]_{\beta'} [S^{-1}]_{\beta'} = [T]_\beta$

$\Rightarrow [T]_{\beta'} = [S]_{\beta'}^{-1} [T]_\beta [S]_{\beta'}$

$= P^{-1} [T]_\beta P$, where $P = [S]_{\beta'}$.

Theorem:- Let V and W be two finite dimensional vector spaces with ordered bases α and β respectively. Let $T: V \to W$ be a L.T and T is invertible. Then $[T^{-1}]_\beta^\alpha = \left([T]_\alpha^\beta\right)^{-1}$

Proof:- Since T is invertible, $\dim V = \dim W$.

Let $\dim V = \dim W = n$.

$T^{-1}: W \to V$.

$TT^{-1} = I_W$ and $T^{-1}T = I_V$.

$\therefore I_{n\times n} = [I_W]_\beta = [TT^{-1}]_\beta = [T]_\alpha^\beta [T^{-1}]_\beta^\alpha$

and $I_{n\times n} = [I_V]_\alpha = [T^{-1}T]_\alpha = [T^{-1}]_\beta^\alpha [T]_\alpha^\beta$

$\therefore [T]_\alpha^\beta [T^{-1}]_\beta^\alpha = [T^{-1}]_\beta^\alpha [T]_\alpha^\beta = I_{n\times n}$

$\therefore [T^{-1}]_\beta^\alpha = \left([T]_\alpha^\beta\right)^{-1}$.

Theorem:- V be a vector space of dimension n over a field F. Then V is isomorphic with F^n. P-253

Problem:- A linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ by

$T(x,y,z) = (x-y, x+2y, y+3z)$, $(x,y,z)\in\mathbb{R}^3$. P-257

Show that T is invertible and determine T^{-1}.

(check) **Problem:-** Let, T be a linear operator on $\mathbb{R}^2$ defined by $T(x_1,x_2) = (-x_2, x_1)$, $\forall (x_1,x_2)\in\mathbb{R}^2$. Show that for any $c\in F$, the operator $(T-cI)$ is invertible.

Solⁿ:- ~~We have {(1,0),(0,1)} is the standard basis of $\mathbb{R}^2$.~~

$\therefore T(1,0) = (0,1)$

$T(0,1) = (-1,0)$

$\therefore [T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\therefore |T| = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 0-(-1) = 1 \neq 0$.

$\therefore$ ~~T is invertible~~

$T - cI : \mathbb{R}^2 \to \mathbb{R}^2$.

Let $(x,y) \in N(T-cI)$

Then $(T-cI)(x,y) = (0,0)$

$\Rightarrow T(x,y) - cI(x,y) = (0,0)$

$\Rightarrow (-y, x) - (cx, cy) = (0,0)$

$\Rightarrow (-y-cx, x-cy) = (0,0)$

$\Rightarrow -y - cx = 0, \quad x - cy = 0 \;\;\text{---(2)}$

$\Rightarrow y + cx = 0 \;\;\text{---(1)}$

$(1)\times y + (2)\times x$,

$y^2 + x^2 = 0$

$\Rightarrow y^2 + c^2y^2 = 0 \quad [\because x = cy]$

$\Rightarrow (1+c^2)y^2 = 0$

$\Rightarrow y^2 = 0 \quad [\because 1 + c^2 = 0 \Rightarrow c^2 = -1 \Rightarrow c \notin \mathbb{R}$, So, it is not possible.]

$\Rightarrow y = 0$.

$\therefore x = 0$.

$\Rightarrow (x,y) = (0,0)$.

$\Rightarrow N(T-cI) = \{(0,0)\}$

$\Rightarrow T-cI$ is one-one.

~~Since,~~ Here $V = \mathbb{R}^2, W = \mathbb{R}^2$.

i.e, dimensions are same and $T-cI$ is one-one.

Thus, $T-cI$ is onto.

Hence, $T-cI$ is invertible.

Theorem:- Let V be a vector space of dimension n over a field F. Then V is isomorphic with F^n.

Proof:- Let $\{\beta_1, \beta_2, \dots, \beta_n\}$ be an ordered basis of V.

Then any vector ξ of V can be expressed as

$\xi = c_1\beta_1 + c_2\beta_2 + \dots + c_n\beta_n$, where $c_1, c_2, \dots, c_n$ are unique scalars in F.

Let us define a mapping $\varphi: V \to F^n$ by

$$\varphi(\xi) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \text{ where } \xi = c_1\beta_1 + c_2\beta_2 + \dots + c_n\beta_n \in V.$$

Let $\alpha = a_1\beta_1 + a_2\beta_2 + \dots + a_n\beta_n \in V$

$\beta = b_1\beta_1 + b_2\beta_2 + \dots + b_n\beta_n \in V$

Now, $\alpha + \beta = (a_1+b_1)\beta_1 + (a_2+b_2)\beta_2 + \dots + (a_n+b_n)\beta_n \in V$

$$\therefore \varphi(\alpha) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad \varphi(\beta) = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\varphi(\alpha+\beta) = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \varphi(\alpha) + \varphi(\beta) \quad \text{---(i)}$$

Let $p \in F$ then $p\alpha \in V$ and $p\alpha = (pa_1)\beta_1 + (pa_2)\beta_2 + \dots + (pa_n)\beta_n$

$$\therefore \varphi(p\alpha) = \begin{pmatrix} pa_1 \\ pa_2 \\ \vdots \\ pa_n \end{pmatrix} = p\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = p\varphi(\alpha) \quad \text{---(ii)}$$

From (i) & (ii), φ is a homomorphism.

To prove that φ is one-to-one, let $\alpha, \beta \in V$ such that $\varphi(\alpha) = \varphi(\beta)$, where $\alpha = a_1\beta_1 + a_2\beta_2 + \dots + a_n\beta_n$

and $\beta = b_1\beta_1 + b_2\beta_2 + \dots + b_n\beta_n$

Now, $\varphi(\alpha) = \varphi(\beta)$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \Rightarrow a_1 = b_1, a_2 = b_2, \dots, a_n = b_n$$

$\Rightarrow \alpha = \beta$.

$\therefore \varphi$ is one-to-one.

To prove that φ is onto, let $\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$ be any element in F^n.

Then $r_1\beta_1 + r_2\beta_2 + \cdots + r_n\beta_n \in V$

and $\varphi(r_1\beta_1 + r_2\beta_2 + \cdots + r_n\beta_n) = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$

So, φ is onto.

Since, φ is both one-to-one and onto then φ is an isomorphism

Since, φ is isomorphism then V is isomorphic to F^n.

Problem: – A linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ by

$T(x,y,z) = (x-y, x+2y, y+3z), (x,y,z) \in \mathbb{R}^3$.

Show that T is invertible and determine T^{-1}.

Sol^n: – Given that $T(x,y,z) = (x-y, x+2y, y+3z)$, $(x,y,z) \in \mathbb{R}^3$.

Let $(x_1, x_2, x_3) \in N(T)$

$\therefore T(x_1, x_2, x_3) = (0,0,0)$

$\Rightarrow \quad x_1 - x_2 = 0, \; x_1 + 2x_2 = 0, \; x_2 + 3x_3 = 0$.

Solving $x_1 = x_2 = x_3 = 0$.

$\therefore N(T) = \{(0,0,0)\}$

Thus, T is one-to-one.

Again, $V = \mathbb{R}^3$, $W = \mathbb{R}^3$.

So, $\dim V = \dim W$.

Since, $T: V \to W$ is one-to-one, so T is onto.

$\therefore$ T is one-to-one and onto so, T is bijective and hence T is invertible.

2nd Part: – Let $T^{-1}(x,y,z) = (a,b,c)$

or, $(x,y,z) = T(a,b,c)$

or, $(x,y,z) = (a-b, a+2b, b+3c)$

$\therefore a-b = x, \quad a+2b = y, \quad b+3c = z$

$2b - (-b) = y - x$

$\Rightarrow \quad b = \frac{1}{3}(y-x)$

$c = \frac{1}{3}(z-b) = \frac{1}{3}(\cancel{z} - \cancel{y} + \cancel{x})$

$= \frac{1}{3}\{z - \frac{1}{3}(y-x)\}$

$= \frac{1}{9}(x - y + 3z)$

$a = x + b$

$= x + \frac{1}{3}y - \frac{1}{3}x$

$= \frac{1}{3}(2x+y)$

$\therefore T^{-1}(x,y,z) = (a,b,c) = \left(\frac{1}{3}(2x+y), \frac{1}{3}(y-x), \frac{1}{9}(x-y+3z)\right)$, $(x,y,z) \in \mathbb{R}^3$.

● **Dual and Double dual:** – Let $V(F)$ be a vector space over a field F. A mapping $f: V \to F$ is said to be linear functional if $f(\alpha x + y) = \alpha f(x) + f(y)$, for all $x, y \in V$ and $\alpha \in F$. The set of all linear functionals from V to F is also a vector space over F, which is denoted by $\hat{V}$.

This space $\hat{V}$ is called dual space of V.
The set of all linear functionals from $\hat{V}$ to F is also a vector space over F.
This vector space is called double dual space of V or dual space of $\hat{V}$. Double dual is denoted by $\hat{\hat{V}}$.

V.V.I ***
Theorem:- Let $\{v_1, v_2, \dots, v_n\}$ be a basis of V. Define mapping $\hat{v}_i : V \to F$ such that $\hat{v}_i(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) = \alpha_i$, $\forall i = 1, 2, \dots, n$. Then $\hat{v}_i$ is linear transformation for all i and $\{\hat{v}_1, \hat{v}_2, \dots, \hat{v}_n\}$ is basis of $\hat{V}$.

Proof:- Let v, v' be two elements of V.
Since, $\{v_1, v_2, \dots, v_n\}$ is a basis of V then

$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \in V$, where $\alpha_1, \alpha_2, \dots, \alpha_n$ are unique scalars in F.
$v' = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n \in V$, where $\beta_1, \beta_2, \dots, \beta_n$ are unique scalars in F.

To show $\hat{v}_i : V \to F$ is well-defined, let
$v = v'$ in V.
Then $\alpha_i = \beta_i$, $\forall i = 1, 2, \dots, n$.
Now, $\hat{v}_i(v) = \hat{v}_i(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) = \alpha_i = \beta_i = \hat{v}_i(v')$, $\forall i = 1, 2$
$\Rightarrow$ $\hat{v}_i$ is well-defined mapping.
Now, $v + v' = (\alpha_1 + \beta_1)v_1 + (\alpha_2 + \beta_2)v_2 + \dots + (\alpha_n + \beta_n)v_n$.
~~To prove $\hat{v}_i$ is one-to-one,~~

$\hat{v}_i(v + v') = \hat{v}_i((\alpha_1 + \beta_1)v_1 + (\alpha_2 + \beta_2)v_2 + \dots + (\alpha_n + \beta_n)v_n)$
$= \alpha_i + \beta_i = \hat{v}_i(v) + \hat{v}_i(v')$

Let $p \in F$ then $pv = (p\alpha_1)v_1 + (p\alpha_2)v_2 + \dots + (p\alpha_n)v_n \in V$.
$\therefore \hat{v}_i(pv) = \hat{v}_i((p\alpha_1)v_1 + (p\alpha_2)v_2 + \dots + (p\alpha_n)v_n)$
$= p\alpha_i = p\,\hat{v}_i(v)$

$\therefore \hat{v}_i$ is a linear transformation, for all $i = 1, 2, \dots, n$.

Let $a_1 \hat{v}_1 + a_2 \hat{v}_2 + \dots + a_n \hat{v}_n = 0$
Then $(a_1 \hat{v}_1 + a_2 \hat{v}_2 + \dots + a_n \hat{v}_n)(v_j) = 0(v_j) = 0$, $\forall j$
$\Rightarrow a_j = 0$, $\forall j = 1, 2, \dots, n$.
$\therefore$ The set of vectors $\{\hat{v}_1, \hat{v}_2, \dots, \hat{v}_n\}$ is L.I.
Let $f \in \hat{V}$ and $f(v_j) = c_j$ $\forall j = 1, 2, \dots, n$.
Now, $(c_1 \hat{v}_1 + c_2 \hat{v}_2 + \dots + c_n \hat{v}_n)(v_j) = c_j$ $\forall j = 1, 2, \dots, n$
$\therefore f(v_j) = (c_1 \hat{v}_1 + c_2 \hat{v}_2 + \dots + c_n \hat{v}_n)(v_j)$
$\Rightarrow f = c_1 \hat{v}_1 + c_2 \hat{v}_2 + \dots + c_n \hat{v}_n$, $\forall j$
$\therefore \{\hat{v}_1, \hat{v}_2, \dots, \hat{v}_n\}$ is basis of $\hat{V}$.

Corollary :- dimension of $\hat{V}$ = dimension of $V = n$

$\therefore \{\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n\}$ dual basis of $\{v_1, v_2, \ldots, v_n\}$

(*) Problem :- Let V be the vector space of all polynomial functions $p: \mathbb{R} \to \mathbb{R}$ which have degree less than or equal to 2. Define three linear functionals on V by $f_1(p) = \int_0^1 p(x)dx$, $f_2(p) = \int_0^2 p(x)dx$ and $f_3(p) = \int_0^{-1} p(x)dx$. Show that $\{f_1, f_2, f_3\}$ basis of $\hat{V}$. $\{f_1, f_2, f_3\}$ be dual basis. Determine a basis of V such that $\{f_1, f_2, f_3\}$ be dual basis.

Soln :- Consider the standard ordered basis of $P_2(\mathbb{R})$ is $\{1, x, x^2\}$.

Now, $f_1, f_2, f_3 \in \hat{V}$, where $V = P_2(\mathbb{R})$.

Let $a_1 f_1 + a_2 f_2 + a_3 f_3 = 0$

Now operating simultaneously on above $1, x, x^2$, we get

$a_1 f_1(1) + a_2 f_2(1) + a_3 f_3(1) = 0$

$\Rightarrow a_1 + 2a_2 - a_3 = 0$

$a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x) = 0$

$\Rightarrow \frac{a_1}{2} + 2a_2 + \frac{1}{2} a_3 = 0 \Rightarrow a_1 + 4a_2 + a_3 = 0$

$a_1 f_1(x^2) + a_2 f_2(x^2) + a_3 f_3(x^2) = 0$

$\Rightarrow \frac{1}{3} a_1 + \frac{8}{3} a_2 - \frac{1}{3} a_3 = 0 \Rightarrow a_1 + 8a_2 - a_3 = 0$.

$$\therefore \begin{pmatrix} 1 & 2 & -1 \\ 1 & 4 & 1 \\ 1 & 8 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 2 & -1 \\ 1 & 4 & 1 \\ 1 & 8 & -1 \end{vmatrix} = 1(-12) - 2(-2) + -1(4) = -12 + 4 - 4 = -12 \neq 0.$$

$\Rightarrow a_1 = a_2 = a_3 = 0$.

$\therefore \{f_1, f_2, f_3\}$ is L.I set.

$\dim \hat{V} = 3$ $[\because \dim P_2(\mathbb{R}) = 3]$

$\therefore \{f_1, f_2, f_3\}$ is basis of $\hat{V}$.

2nd Part :- Let $\{p_1(x), p_2(x), p_3(x)\}$ be a basis of V s.t. $\{f_1, f_2, f_3\}$ is dual basis.

Now, $f_1(p_1) = 1$, $f_2(p_1) = 0$, $f_3(p_1) = 0$.

Let $p_1(x) = c_0 + c_1 x + c_2 x^2$.

Now, $f_2(p_1) = 0$

$\Rightarrow \int_0^2 (c_0 + c_1 x + c_2 x^2) dx = 0$

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$$\Rightarrow \left. c_0x + c_1\frac{x^2}{2} + c_2\frac{x^3}{3}\right]_{x=2} = 0$$

and $f_3(p_1) = 0$

$$\Rightarrow \left. c_0x + c_1\frac{x^2}{2} + c_2\frac{x^3}{3}\right]_{x=-1} = 0$$

$$\therefore c_0x + c_1\frac{x^2}{2} + c_2\frac{x^3}{3} = \alpha x(x-2)(x+1) \quad \text{---} (1)$$

Now, $f_1(p_1) = 1$

$$\Rightarrow \left. c_0x + c_1\frac{x^2}{2} + c_2\frac{x^3}{3}\right]_{x=1} = 1$$

$$\Rightarrow \alpha \cdot 1(1-2)(1+1) = 1 \quad [\text{from } (1)]$$

$$\Rightarrow \alpha = -\frac{1}{2}$$

$$\therefore c_0x + c_1\frac{x^2}{2} + c_2\frac{x^3}{3} = -\frac{1}{2}x(x-2)(x+1)$$

$$= -\frac{1}{2}\{(x^2-2x)(x+1)\}$$

$$= -\frac{1}{2}x^3 + \frac{1}{2}x^2 + x$$

Comparing, we get

$$c_0 = 1, \quad c_1 = 1, \quad c_2 = -\frac{3}{2}$$

$$\therefore p_1(x) = 1 + x - \frac{3}{2}x^2$$

Now, $f_1(p_2) = 0$, $f_2(p_2) = 1$, $f_3(p_2) = 0$

Let $p_2 = d_0 + d_1x + d_2x^2$

$\therefore f_1(p_2) = 0$

$$\Rightarrow \int_0^1 (d_0 + d_1x + d_2x^2)\,dx = 0$$

$$\Rightarrow \left. d_0x + d_1\frac{x^2}{2} + d_2\frac{x^3}{3}\right]_{x=1} = 0$$

$f_3(p_2) = 0$

$$\Rightarrow \left. d_0x + d_1\frac{x^2}{2} + d_2\frac{x^3}{3}\right]_{x=-1} = 0$$

$$\therefore d_0x + d_1\frac{x^2}{2} + d_2\frac{x^3}{3} = \beta x(x-1)(x+1) \quad \text{---} (2)$$

Again, $f_2(p_2) = 1$

$$\Rightarrow \left. d_0x + d_1\frac{x^2}{2} + d_2\frac{x^3}{3}\right]_{x=2} = 1$$

$$\Rightarrow \beta \cdot 2(2-1)(2+1) = 1 \quad [\text{from } (2)]$$

$$\Rightarrow \beta = \frac{1}{6}$$

$$\therefore d_0x + d_1\frac{x^2}{2} + d_2\frac{x^3}{3} = \frac{1}{6}x(x^2-1) = \frac{1}{6}x^3 - \frac{1}{6}x$$

Comparing, we get

$$d_0 = -\frac{1}{6}, \quad d_1 = 0, \quad d_2 = \frac{1}{2}$$

$$\therefore p_2(x) = -\frac{1}{6} + \frac{1}{2}x^2$$

Now, ~~$f_3(p_3)=0, f_3(p_2)$~~

$f_1(p_3)=0,\ f_2(p_3)=0,\ f_3(p_3)=1.$

Let $p_3 = e_0 + e_1 x + e_2 x^2$

$\therefore\ f_1(p_3)=0$

$\Rightarrow \int_0^1 (e_0 + e_1 x + e_2 x^2)\,dx = 0$

$\Rightarrow \left. e_0 x + e_1 \frac{x^2}{2} + e_2 \frac{x^3}{3} \right]_{x=1} = 0$

$f_2(p_3)=0$

$\Rightarrow \left. e_0 x + e_1 \frac{x^2}{2} + e_2 \frac{x^3}{3} \right]_{x=2} = 0$

$\therefore\ e_0 x + e_1 \frac{x^2}{2} + e_2 \frac{x^3}{3} = \gamma x(x-1)(x-2)$ —— (3)

Again, $f_3(p_3)=1$

$\Rightarrow \left. e_0 x + e_1 \frac{x^2}{2} + e_2 \frac{x^3}{3} \right]_{x=-1} = 1.$

$\Rightarrow \gamma(-1)(-1-1)(-1-2) = 1$ [from (3)]

$\Rightarrow \gamma = -\frac{1}{6}$

$\therefore\ e_0 x + e_1 \frac{x^2}{2} + e_2 \frac{x^3}{3} = -\frac{1}{6}(x^2-x)(x-2)$

$= -\frac{1}{6}(x^3 - 3x^2 + 2x)$

Comparing, $e_0 = -\frac{1}{3},\ e_1 = 1,\ e_2 = -\frac{1}{2}$

$\therefore\ p_3(x) = -\frac{1}{3} + x - \frac{1}{2}x^2.$

$\therefore \left\{ 1 + x - \frac{3}{2}x^2,\ -\frac{1}{6} + \frac{1}{2}x^2,\ -\frac{1}{3} + x - \frac{1}{2}x^2 \right\}$ is the basis of V s.t. $\{f_1, f_2, f_3\}$ is dual basis.

Annihilator:- Let $V(F)$ be a vector space and W be a subset of V.

Now, $A(W) = \{ f \in \hat{V} : f(\omega) = 0,\ \forall\, \omega \in W \}$

Then $A(W)$ is a subspace of $\hat{V}$.

$A(W)$ is called annihilator of W.

Theorem:- Let V be a finite dimensional vector space and W, a subspace of V. Then $\dim A(W) = \dim V - \dim W$.

Proof:- Let $\{\omega_1, \omega_2, \ldots, \omega_m\}$ be a basis of W.

It can be extend to form a basis of V.

Let $\{\omega_1, \omega_2, \ldots, \omega_m, v_{m+1}, v_{m+2}, \ldots, v_n\}$ be a basis of V.

Let $\{f_1, f_2, \ldots, f_m, f_{m+1}, f_{m+2}, \ldots, f_n\}$ be the corresponding dual basis. ~~of W~~

Now, $f_i(\omega_j) = 0$, $\forall\, i = m+1, m+2, \ldots, n$

and $j = 1, 2, \ldots, m$

$\Rightarrow f_i \in A(W)$, $\forall i = m+1, m+2, \ldots, n$.

We show that $\{f_{m+1}, f_{m+2}, \ldots, f_n\}$ is a basis of $A(W)$.

Let $\alpha_1 f_{m+1} + \alpha_2 f_{m+2} + \cdots + \alpha_n f_n = 0$

$\Rightarrow (\alpha_1 f_{m+1} + \alpha_2 f_{m+2} + \cdots + \alpha_n f_n)(v_k) = 0$, $\forall k = m+1, m+2, \ldots, n$

$\Rightarrow \alpha_k f_k(v_k) = 0$

$\Rightarrow \alpha_k = 0$, $\forall k = m+1, m+2, \ldots, n$.

$\therefore \{f_{m+1}, f_{m+2}, \ldots, f_n\}$ is L.I set.

Let $f \in A(W)$.

Then $f(\omega) = 0, \forall \omega \in W$ and $f \in \hat{V}$

Now, $f \in \hat{V}$

$\Rightarrow f = \beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_n f_n$

$\Rightarrow 0 = f(\omega_j) = \beta_j f_j(\omega_j)$, for all $j = 1, 2, \ldots, m$

$\Rightarrow \beta_j = 0$, for all $j = 1, 2, \ldots, m$.

$\therefore f = \beta_{m+1} f_{m+1} + \beta_{m+2} f_{m+2} + \cdots + \beta_n f_n$.

$\Rightarrow$ ~~f span~~ $\{f_{m+1}, f_{m+2}, \ldots, f_n\}$ spans $A(W)$.

Thus, $\{f_{m+1}, f_{m+2}, \ldots, f_n\}$ is a basis of $A(W)$.

Hence, $\dim A(W) = n - m = \dim V - \dim W$.

Hence, the proof.

Problem :- Let W be a subspace of $\mathbb{R}^5$ spanned by the vectors $\alpha_1 = (2, -2, 3, 4, -1)$, $\alpha_2 = (-1, 1, 2, 5, 2)$, $\alpha_3 = (0, 0, -1, -2, 3)$ and $\alpha_4 = (1, -1, 2, 3, 0)$. Describe $A(W)$.

Solⁿ :- Let $f \in A(W)$

Then $f(\omega) = 0$, $\forall \omega \in W$

and $f \in \hat{V}$, where $V = \mathbb{R}^5$.

Then $f(\alpha_i) = 0$, $\forall i = 1, 2, 3, 4$.

Let $\{v_1, v_2, v_3, v_4, v_5\}$ be the standard ordered basis of $\mathbb{R}^5$.

Now, $f \in \hat{V}$

$\Rightarrow f = c_1 \hat{v}_1 + c_2 \hat{v}_2 + c_3 \hat{v}_3 + c_4 \hat{v}_4 + c_5 \hat{v}_5$

$= \sum_{i=1}^{5} c_i \hat{v}_i$

$$\therefore f(x_1,x_2,x_3,x_4,x_5) = \sum_{i=1}^{5} c_i \hat{v}_i (x_1,x_2,x_3,x_4,x_5)$$

$$= \sum_{i=1}^{5} c_i \hat{v}_i (x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 + x_5v_5)$$

$$= c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 .$$

, where $c_i \in \mathbb{R}$ for $i = 1(1)5$.

$$\therefore f(2,-2,3,4,-1) = 2c_1 - 2c_2 + 3c_3 + 4c_4 - c_5 = 0$$

$$f(-1,+1,2,5,2) = -c_1 + c_2 + 2c_3 + 5c_4 + 2c_5 = 0$$

$$f(0,0,-1,-2,3) = 0c_1 + 0c_2 - c_3 - 2c_4 + 3c_5 = 0$$

$$f(1,-1,2,3,0) = c_1 - c_2 + 2c_3 + 3c_4 + 0c_5 = 0.$$

$$\therefore \begin{pmatrix} 2 & -2 & 3 & 4 & -1 \\ -1 & +1 & 2 & 5 & 2 \\ 0 & 0 & -1 & -2 & 3 \\ 1 & -1 & 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow (1)$$

$$\begin{pmatrix} 2 & -2 & 3 & 4 & -1 \\ -1 & 1 & 2 & 5 & 2 \\ 0 & 0 & -1 & -2 & 3 \\ 1 & -1 & 2 & 3 & 0 \end{pmatrix} \xrightarrow[R_4 - \frac{1}{2}R_1]{R_2 + R_4} \begin{pmatrix} 2 & -2 & 3 & 4 & -1 \\ 0 & 0 & 4 & 8 & 2 \\ 0 & 0 & -1 & -2 & 3 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & -1 & \frac{3}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 4 & 8 & 2 \\ 0 & 0 & -1 & -2 & 3 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix} \xrightarrow[R_3 + 2R_4]{R_1 - \frac{3}{8}R_2} \begin{pmatrix} 1 & -1 & 0 & -1 & -\frac{5}{4} \\ 0 & 0 & 4 & 8 & 2 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}$$

$$\xrightarrow{R_4 - \frac{1}{8}R_2} \begin{pmatrix} 1 & -1 & 0 & -1 & -\frac{5}{4} \\ 0 & 0 & 4 & 8 & 2 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \xrightarrow[\frac{1}{4}R_4]{\frac{1}{2}R_2} \begin{pmatrix} 1 & -1 & 0 & -1 & -\frac{5}{4} \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

$$\xrightarrow[R_4 - \frac{1}{4}R_3]{R_2 - R_3} \begin{pmatrix} 1 & -1 & 0 & -1 & -\frac{5}{4} \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_1 + \frac{5}{4}R_3]{\frac{1}{2}R_2} \begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ (1) becomes,

$$\begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore \quad c_1 - c_2 - c_4 = 0$

$c_3 + 2c_1 = 0 \Rightarrow c_3 = -2c_1$

$c_5 = 0$

Let $c_1 = a,\ c_2 = b$.

$\therefore\ c_4 = a - b$

$\therefore\ c_3 = -2(a-b) = -2a + 2b$.

$\therefore\ f(x_1, x_2, x_3, x_4, x_5) = ax_1 + bx_2 + (-2a+2b)x_3 + (a-b)x_4$

$= a(x_1 - 2x_3 + x_4) + b(x_2 + 2x_3 - x_4)$

Now, for $a=1$, $b=0$; ~~Let~~ $\Rightarrow f_1 = x_1 - 2x_3 + x_4$; for $a=0$, $b=1$, $\Rightarrow f_2 = x_2 + 2x_3 - x_4$.

$\therefore\ f = af_1 + bf_2$.

To show $\{f_1, f_2\}$ is L.I.,

let $\alpha_1 f_1 + \alpha_2 f_2 = 0$.

Operate v_1, $\alpha_1 f_1(v_1) + \alpha_2 f_2(v_1) = 0$

$\Rightarrow \alpha_1 \cdot 1 + \alpha_2 \cdot 0 = 0 \Rightarrow \alpha_1 = 0$.

Operate v_2, $\alpha_1 f_1(v_2) + \alpha_2 f_2(v_2) = 0$

$\Rightarrow \alpha_1 \cdot 0 + \alpha_2 \cdot 1 = 0$

$\Rightarrow \alpha_2 = 0$

$\therefore\ \{f_1, f_2\}$ is L.I.

~~Diagonal~~ DIAGONALISATION OF LINEAR OPERATOR

Let $V(F)$ be a vector space and T ~~is~~ be a linear operator on V. Then T is said to be diagonalisable if $\exists$ ordered basis β of V s.t. the matrix $[T]_\beta$ is a diagonal matrix.

Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be ordered basis of V. Then

$$T(v_j) = \sum_{i=1}^{n} a_{ij} v_i \quad , \quad j = 1, 2, \ldots, n.$$

Now, if $[T]_\beta$ is diagonalisable, then $T(v_j) = a_{jj} v_j$, $j = 1, 2, \ldots$

Eigen value and eigen vector:-

Let $V(F)$ be a vector space and T be a linear operator on V. $0 \neq v \in V$ is said to be eigen ~~value~~ vector of T if $T(v) = cv$ and c is called corresponding eigen value of T.

~~Let T: C~~

Ex:- Let, $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ by $T(f) = f'$.

$C^\infty(\mathbb{R}) \Rightarrow$ set of all infinite times derivable

Now, let f is an eigen vector of T.

$\hookrightarrow \quad T(f) = \lambda f$ or $f' = \lambda f$

or $\dfrac{f'}{f} = \lambda$

or $\log f = \lambda x + \log a$

or, $f = a e^{\lambda x}$

$\therefore$ $f = e^{\lambda x}$ is eigen vector of T.

and λ is the eigen value of T.

$$T(f) = T(e^{\lambda x}) = \lambda e^{\lambda x} = \lambda f$$

Definition:- Let T be a linear operator on a finite dimensional vector space V of dimension n and β be ordered basis of V. Then the characteristic polynomial $f(t)$ of T is the characteristic polynomial of $[T]_\beta$.

$$\therefore \; f(t) = \det([T]_\beta - \lambda I_n)$$

Ex:- Let $T: P_2(R) \to P_2(R)$ defined by $T(f(x)) = f(x) + (x+1)f'(x)$.

Soln:- Let $\beta = \{1, x, x^2\}$ is the standard basis of $P_2(\mathbb{R})$.

Then $[T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

$\therefore \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = -(\lambda-1)(\lambda-2)(\lambda-3)$

$\therefore \lambda = 1, 2, 3$ are eigen values of A as well as T.

$T(f) = f$ i.e, $f(x) + (x+1)f'(x) = f(x)$

$T(f) = 2f$ i.e., $f(x) + (x+1)f'(x) = 2f(x)$

$T(f) = 3f$ i.e, $f(x) + (x+1)f'(x) = 3f(x)$

⊛ Show that characteristic equation of similar matrix is same.

↳ Let A and B be two similar matrix

then $A = PBP^{-1}$

$|A| = |PBP^{-1}| = |P||B||P^{-1}| = |B|$

$|A - \lambda I| = |PBP^{-1} - \lambda I| = |PBP^{-1} - P\lambda P^{-1}|$

$= |P(B - \lambda I)P^{-1}|$

$= |P|\,|B - \lambda I|\,|P^{-1}|$

$= |B - \lambda I|$

Hence, characteristic equation of A and B are same.

$\boxed{\text{NOTE}}$:-

Let β and β' be two ordered basis of V. and T be a linear operator on V.

Then we know that the matrix $[T]_\beta$ and $[T]_{\beta'}$ are similar

Hence, characteristic equations are same for both matrix

Theorem:- Let $\dim V = n$ and T be a linear operator on V. Let $\{v_1, v_2, \dots, v_k\}$ be eigen vector of T corresponding ~~two~~ to distinct eigen values $c_1, c_2, \dots, c_k$. Then $\{v_1, v_2, \dots, v_k\}$ ~~is~~ ~~are~~ L.I.

Proof:- Let $T(v_i) = c_i v_i$; $i = 1, 2, \dots, k$

$T^2(v_i) = T(T(v_i)) = T(c_i v_i) = c_i^2 v_i$

$T^r(v_i) = c_i^r v_i \,;\, r \geq 1$

Let $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$ ——— (1)

Now operating $T, T^2, \ldots, T^{k-1}$ on the above equation

$$\alpha_1 c_1 v_1 + \alpha_2 c_2 v_2 + \cdots + \alpha_k c_k v_k = 0$$

$$\vdots$$

$$\alpha_1 c_1^{k-1} v_1 + \alpha_2 c_2^{k-1} v_2 + \cdots + \alpha_k c_k^{k-1} v_k = 0$$

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ c_1 & c_2 & \cdots & c_k \\ \vdots & & & \vdots \\ c_1^{k-1} & c_2^{k-1} & \cdots & c_k^{k-1} \end{pmatrix} \begin{pmatrix} \alpha_1 v_1 \\ \alpha_2 v_2 \\ \vdots \\ \alpha_k v_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Since all $c_1, c_2, \ldots, c_k$ are distinct,

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ c_1 & c_2 & \cdots & c_k \\ \vdots & \vdots & & \vdots \\ c_1^{k-1} & c_2^{k-1} & \cdots & c_k^{k-1} \end{vmatrix} \neq 0$$

$\therefore \alpha_i v_i = 0, \ \forall i$

Now, $\alpha_i = 0 \quad [\because v_i \neq 0]$

Hence, $\{v_1, v_2, \ldots, v_k\}$ are L.I.

Problem:- Let $A = (a_{ij})_{n\times n}$ s.t. (i) $\sum_j a_{ij} = 1, \forall i$

(ii) $\sum_i a_{ij} = 1, \forall j$

Prove that ~~(i) is~~ in each case 1 is eigen value of A.

Ans:- Let T be a linear operator on V and $\beta = \{v_1, v_2, \ldots, v_n\}$ be ordered basis of V.

$$T(v_1) = a_{11} v_1 + a_{21} v_2 + \cdots + a_{n1} v_n$$

$$T(v_2) = a_{12} v_1 + a_{22} v_2 + \cdots + a_{n2} v_n$$

$$\cdots$$

$$T(v_n) = a_{1n} v_1 + a_{2n} v_2 + \cdots + a_{nn} v_n$$

$T(v_1+v_2+\cdots+v_n) = T(v_1)+T(v_2)+\cdots+T(v_n)$

$= (a_{11}+a_{12}+\cdots+a_{1n})v_1 + \cdots$

$+ (a_{n1}+a_{n2}+\cdots+a_{nn})v_n$

$= (v_1+v_2+\cdots+v_n)$ [using (i)]

Now, $(v_1+v_2+\cdots+v_n) \in V$

$\therefore 1$ is eigen ~~vector~~ value of A.

Taking transpose of A i.e, A';

$\therefore 1$ is eigen value of A'.

Thus, $\det(A'-I)=0$

Or, $\det(A-I)'=0$

$\therefore \det(A-I)=0$

$\therefore 1$ is ~~char~~ eigen value of A.

Problem:- Let V be the vector space of all real valued continuous ~~constant~~ function. Define $T: V \to V$ by $T(f(x)) = \int_0^x f(t)\,dt$. Show that T has no eigen value.

Ans:- By deffinition, $T(f) = cf$ (Assuming c is eigen value)

$\Rightarrow \int_0^x f(t)\,dt = cf(x)$

$\Rightarrow f(x) = cf'(x)$

$\Rightarrow \frac{f'(x)}{f(x)} = \frac{1}{c}$

int, $\log f(x) = \frac{x}{c} + \log a$

$\therefore f(x) = ae^{\frac{x}{c}}$

Now, $T(f) = cf$

$\Rightarrow T(ae^{\frac{x}{c}}) = cf$

$\Rightarrow a\int_0^x e^{\frac{t}{c}}dt = cae^{\frac{x}{c}}$

$\Rightarrow ac(e^{\frac{x}{c}}-1) = cae^{\frac{x}{c}}$

$\Rightarrow e^{\frac{x}{c}}-1 = e^{\frac{x}{c}}$

$\Rightarrow 1=0$, which is a contradiction.

Hence, T has no eigen value.

Minimal Polynomial :- Let T be a linear operator on a finite dimensional vector space $V(F)$. A polynomial $p(t)$ is said to be minimal polynomial if it is monic polynomial of least positive degree such that $p(T) = T_0$.

Theorem-1 :- Let $p(t)$ be a minimal polynomial of a linear operator T on a finite dimensional vector space V

(a) For any polynomial $g(t)$, if $g(T) = T_0$, then $p(t) | g(t)$

(b) The minimal polynomial is unique.

Proof :- $p(T) = T_0$

By the division algorithm, $\exists$ $q(t)$ and $r(t)$ such that

$$g(t) = p(t)q(t) + r(t)$$

, where $r(t) = 0$ or degree $r(t) <$ degree $p(t)$

~~let deg~~

$$g(T) = p(T)q(T) + r(T)$$

Or, $T_0 = T_0\, q(T) + r(T)$

Or, $r(T) = T_0$

Now, suppose degree $r(t) <$ degree $p(t)$ such that

$$r(T) = T_0 .$$

Then this leads to the contradiction.

$$\therefore r(t) = 0$$

$\therefore$ $p(t)$ divides $g(t)$.

(b) Let $p_1(t)$ and $p_2(t)$ be two minimal polynomial.

Then $p_1(T) = T_0$, $p_2(T) = T_0$

Then $p_1(t) | p_2(t)$ i.e., $p_2(t) = c\, p_1(t)$, $c \in F$.

Now, $c = 1$, they are monic polynomial.

$\therefore$ $p_1(t) = p_2(t)$.

Theorem-2:- Let T be a linear operator on a finite dimensional vector space V and $p(t)$ be the minimal polynomial of T. A scalar λ is an eigen value of T if and only if $p(\lambda)=0$. Hence the characteristic polynomial and minimal polynomial have same zeros.

Proof:- Let $f(t)$ be the characteristic polynomial of T.

Then $f(t) = p(t)\,q(t)$ $\quad$ $\because f(T) = T_0$

Let, $p(\lambda)=0$.

$f(\lambda) = p(\lambda)\cdot q(\lambda) = 0$

$\therefore f(\lambda)=0$

$\therefore \lambda$ is the eigen value of T.

Let λ is an eigen value of T, i.e., $f(\lambda)=0$.

$\therefore Tv = \lambda v$ for $v \in V$.

$p(T)v = p(\lambda)v$

Now, $p(T) = T_0$.

$\therefore p(\lambda)v = 0$

$\Rightarrow p(\lambda)=0$ $[\because v \neq 0]$

Hence, characteristic polynomial and minimal polynomial have same zeros.

Theorem 3:- Let $A = [T]_\beta$ and $p(t)$ be the minimal polynomial of T. Then $p(t)$ is also the minimal polynomial of A.

Proof:- Now we have,

$$[p(T)]_\beta = p(A) \quad \because [T^n]_\beta = A^n$$

$\because p(T) = T_0$

$\therefore p(A) = 0$.

Let $q(t)$ be the minimal polynomial of A.

$\therefore q(A)=0$ and we have $p(A)=0$.

$p(t) = q(t)\cdot h(t) + r(t)$.

$r(t)=0$ or degree $r(t) <$ degree $q(t)$.

$\Rightarrow r(t)=0$

i.e., $p(t) = q(t)\cdot h(t)$

$\Rightarrow q(t) \mid p(t)$ ——— (i)

We have, $[q(T)]_\beta = q(A)$.

Now, $q(t)$ is the minimal polynomial.

So, $q(A)=0$, $q(T)=T_0$.

But we have, $p(T)=T_0$.

By division algorithm, $\exists$ $h_1(t)$ and $r_1(t)$ such that

$$q(t) = p(t)\cdot h_1(t) + r_1(t), \quad r_1(t)=0 \text{ or } \text{degree } r_1(t) < \text{degree}$$

$\Rightarrow r_1(t) = 0$.

$\therefore p(t) \mid q(t)$ ——— (ii)

$\therefore p(t) = q(t)$

$\boxed{\text{NOTE}}$:- Similar matrices have same minimal polynomial

(Thm-2nd) **Corollary-2** :- Let T be a linear operator on a finite dimensional vector space V with minimal polynomial $p(t)$ and characteristic polynomial $f(t)$. Suppose

$$f(t) = (t-\lambda_1)^{n_1}(t-\lambda_2)^{n_2}\cdots\cdots(t-\lambda_n)^{n_n}$$

, where $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are the eigen values of T.

Then $\exists$ integers $m_1, m_2, \ldots\ldots, m_n$ such that $1 \le m_i \le n_i$, $\forall i$

and $p(t) = (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\cdots\cdots(t-\lambda_n)^{m_n}$

Ex:- $f(t) = (t-2)^3(t-3)^2$

Possibilities:- $p_1(t) = (t-2)(t-3)$

$p_2(t) = (t-2)(t-3)^2$

$p_3(t) = (t-2)^2(t-3)$

$p_4(t) = (t-2)^2(t-3)^2$

$p_5(t) = f(t)$

$p_6(t) = (t-2)^3(t-3)$

~~Com~~

Problem :- Compute the minimal polynomial of

$$A = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

$\hookrightarrow$ $f(t) = -(t-2)^2(t-3)$.

The possibilites of minimal polynomial are

$p_1(t) = -(t-2)(t-3)$

$p_2(t) = -(t-2)^2(t-3)$

Here $p_1(A) = 0$

$\therefore -(t-2)(t-3) = p_1(t)$ is the minimal polynomial.

Ex-2:- Find minimal polynomial of $T: \mathbb{R}^2 \to \mathbb{R}^2$ by

$T(a, b) = (2a + 5b, 6a + b)$

$\hookrightarrow \quad [T]_\beta = \begin{pmatrix} 2 & 5 \\ 6 & 1 \end{pmatrix}$

Here $f(t) = (t-7)(t+4)$ is the characteristic polynomial

$\therefore p(t) = (t-7)(t+4)$ is the minimal polynomial.

Problem-3:- Let a, b, c be elements of field F and

$A = \begin{bmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix}$. Show that characteristic polynomial of A and minimal polynomial are same.

Soln:- $|A - \lambda I| = \begin{vmatrix} -\lambda & 0 & c \\ 1 & -\lambda & b \\ 0 & 1 & a-\lambda \end{vmatrix} = 0$

$\Rightarrow -\lambda(-a\lambda + \lambda^2 - b) + c(1) = 0$

$\Rightarrow -\lambda^3 + a\lambda^2 + b\lambda + c = 0$

$\Rightarrow \lambda^3 - a\lambda^2 - b\lambda - c = 0$

$\therefore$ The characteristic polynomial is

$f(t) = t^3 - at^2 - bt - c$

Let $p(t)$ be the minimal polynomial of A.

$\therefore \quad p(t) \mid f(t) \quad$ and

let degree of $p(t) = 1$ i.e., let $p(t) = t + \alpha$, $\alpha \in F$.

By definition, $p(A) = 0$

i.e., $A + \alpha I = 0$

$\Rightarrow \begin{bmatrix} \alpha & 0 & c \\ 1 & \alpha & b \\ 0 & 1 & a+\alpha \end{bmatrix} = 0$

, which is a contradiction $[\because 1 \neq 0]$.

Let degree of $p(t) = 2$ and let

$p(t) = t^2 + \beta t + \alpha$

By definition, $p(A) = 0$

i.e., $\begin{bmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix}^2 + \beta \begin{bmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix} + \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} = 0$

$$\Rightarrow \begin{bmatrix} 0 & c & ac \\ 0 & b & c+ab \\ 1 & a & b+a^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & c\beta \\ \beta & 0 & b\beta \\ 0 & \beta & a\beta \end{bmatrix} + \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} \alpha & c & c(a+\beta) \\ \beta & b+\alpha & c+b(a+\beta) \\ 1 & a+\beta & b+\alpha+a(a+\beta) \end{bmatrix} = 0$$

, which is a contradiction as $1 \neq 0$.

$\therefore$ Degree of $p(t) = 3$.

$\because$ $p(t) \mid f(t)$

$\therefore f(t) = c\, p(t)$, $c \in F$

$\because$ $p(t)$ is monic, then $c = 1$.

$\therefore f(t) = p(t)$.

Theorem:- Let $A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$, where B, C are square matrices. Then minimal polynomial of A is l.c.m of the minimal polynomials $q(x)$, $r(x)$ of B and C, resp.

Proof:- Let the minimal polynomial of A is $p(x)$.

Now, $p(A) = \begin{pmatrix} p(B) & 0 \\ 0 & p(C) \end{pmatrix} = 0$

$\therefore p(B) = 0$ and $p(C) = 0$.

$q(x) \mid p(x)$ and $r(x) \mid p(x)$ ——— (1)

Let $q(x) \mid f(x)$ and $r(x) \mid f(x)$ ——— (2)

Now, $f(A) = \begin{pmatrix} f(B) & 0 \\ 0 & f(C) \end{pmatrix}$

Now, $q(x) \mid f(x)$

then $f(B) = 0$ $\Big[\because f(x) = q(x) \cdot h(x)$, for any polynomial $h(x)$

$\therefore f(B) = q(B) \cdot h(B)$

$= 0$ $[\because q(x)$ is minimal polynomial of B $\therefore q(B) = 0$

and $r(x) \mid f(x)$

then $f(C) = 0$

$\therefore f(A) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

Now, $p(x)$ is minimal polynomial of A.

$\therefore p(x) \mid f(x)$ ——— (3)

So, from (1), (2) & (3) the conclusion is that

$p(x) = \text{l.c.m}\{q(x), r(x)\}$

H/W Let $A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Find the minimal polynomial of the matrix

$$\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}.$$

Eigen space:- Let T be a linear operator on a finite dimensional vector space V. Let $c_1, c_2, \ldots, c_k$ be eigen value of T. Now,

$$W_{c_i} = \{v_j \in V : Tv_j = c_i v_j\}$$
$$= \{v_j \in V : (T - c_i I)v_j = 0\}$$
$$= \ker(T - c_i I)$$

W_{c_i} is called eigen space for c_i.

Theorem:- Let T be a linear operator on a finite dimensional vector space $V(F)$. Let $c_1, c_2, \ldots, c_k \in F$ be distinct eigen values of T and let W_i be the eigen space corresponding to eigen value c_i, $i = 1, 2, \ldots, k$. Suppose $\beta_1, \beta_2, \ldots, \beta_k$ are basis of $W_1, W_2, \ldots, W_k$. Then $\beta = \{\beta_1, \beta_2, \ldots, \beta_k\}$ is a basis of $W = W_1 + W_2 + \cdots + W_k$ and $\dim W = \dim W_1 + \dim W_2 + \cdots + \dim W_k$.

Theorem:- Let T be a linear operator on a finite dimensional vector space $V(F)$. Then T is diagonalisable if and only if $\dim V = \dim W_1 + \dim W_2 + \cdots + \dim W_k$.

Theorem:- Let T be a linear operator on a finite dimensional vector space $V(F)$. Let $p(x)$ be the minimal polynomial of T. Then T is diagonalisable if and only if

$$p(x) = (x - c_1)(x - c_2) \cdots (x - c_k)$$

, where $c_1, c_2, \ldots, c_k \in F$ are distinct eigen values.

Proof:- Let T is diagonalisable and ~~$p(x)$~~

$$p'(x) = (x - c_1)(x - c_2) \cdots (x - c_k)$$ be a

polynomial. Then $\exists$ a basis $\beta = \{v_1, v_2, \ldots, v_k\}$ of eigen vectors of T in V.

Now, consider $c_i \in F$ be an eigen value of T.

Then $\exists$ some eigen vectors $v_j \in V$, ~~$j = 1, 2, \ldots, k$~~ such that

$T(v_j) = c_i v_j$, ~~$j=1,3,\dots,k$~~

Or, $W_{c_i} = \{v_j \in V : T(v_j) = c_i v_j\}$

Or, $W_{c_i} = \{v_j \in V : (T - c_i I) v_j = 0\}$

$= \ker(T - c_i I)$

Now, $p(x) = (x-c_1)(x-c_2)\cdots(x-c_k)$

$= (x-c_i)\, q(x)$

Or, $p(T) = (T - c_i I)\, q(T)$

$p(T) v_j = (T - c_i I)(v_j)\, q(T)$, where $v_j \in V$ be an eigen vector.

$= 0$

$\therefore p(T) v_j = 0, \ \forall v_j$, $j = 1,3,\dots,k$.

$\therefore p(T) = T_0 \quad [\because v_j \neq 0]$

$\therefore p(x)$ is minimal polynomial.

~~Theorem~~ Let $p(x) = (x-c_1)(x-c_2)\cdots(x-c_k)$ be a minimal polynomial of T.

Let $f_i(x) = \dfrac{p(x)}{x - c_i} = \dfrac{\cancel{(x-c_2)\cdots}(x-c_k)}{}$
$(x-c_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_k)$, $i = 1,3,\dots,k$.

Then $g.c.d\{f_1, f_2, \dots, f_k\} = 1$ $[\because c_1, c_2, \dots, c_k$ all are distinct$]$

$\therefore \exists$ a polynomial $g_1(x), g_2(x), \dots, g_k(x) \in F(x)$ s.t.

$f_1 g_1 + f_2 g_2 + \cdots + f_k g_k = 1$

$\therefore f_1(T) g_1(T) + \cdots + f_k(T) g_k(T) = I$

$\boxed{F(x) = \text{set of all polynomials with coefficient in } F}$

$\Rightarrow f_1(T) g_1(T) v + \cdots + f_k(T) g_k(T) v = I(v) = v$; for $v \in V$. —— (1)

Now, $f_1(T) = \dfrac{p(T)}{T - c_1 I}$

$\therefore (T - c_1 I) f_1(T) = \cancel{(T - c_1 I)}\cdots\cancel{(T - c_k I)} = p(T)$
$= T_0$

$\therefore f_1(T) \in \ker(T - c_1 I) = W_{c_1}$

Similarly, $f_i(T) \in \ker(T - c_i I) = W_{c_i}$

Then $f_i(T) g_i(T) \in W_{c_i}$; $\forall i$

$\left[\begin{array}{l} \because (T - c_i I) f_i(T) = T_0 \\ (T - c_i I) f_i(T) g_i(T) = T_0 \\ \Rightarrow f_i(T) g_i(T) \in W_{c_i}, \ \forall i \end{array}\right]$

$\therefore$ From (1), for $v \in V$

$$v \in W_{c_1} + W_{c_2} + \cdots + W_{c_k}$$

$$\therefore V = W_{c_1} + W_{c_2} + \cdots + W_{c_k}$$

$$\therefore \dim V = \dim W_{c_1} + \dim W_{c_2} + \cdots + \dim W_{c_k}$$

Theorem:- Let T be linear operator on an n-dimensional vector space V, and suppose that T has n distinct characteristic values. Then T is diagonalisable.

Proof:- Let $c_1, c_2, \ldots, c_n$ be eigen values of T and $v_1, v_2, \ldots, v_n$ corresponding eigen vector.

$$T(v_i) = c_i v_i, \forall i$$

$$(T - c_i I)(v_i) = 0$$

$$\therefore v_i \in \ker(T - c_i I) = W_{c_i}, \quad v_i \neq 0.$$

$$\therefore \dim W_{c_i} \geq 1 \; ; \forall i.$$

$$\therefore \dim W_{c_1} + \dim W_{c_2} + \cdots + \dim W_{c_n} \geq n = \dim V$$

$$\therefore \dim V = \dim W_{c_1} + \dim W_{c_2} + \cdots + \dim W_{c_n}$$

Hence, T is diagonalisable.

Definition:- Let $T \in L(V)$ and $\dim V = n$. β be an ordered basis of $V(F)$. Then the standard representation of V w.r.to β is the function $\varphi_\beta : V \longrightarrow F^n$ defined by $\varphi_\beta(x) = [x]_\beta$, where $[x]_\beta$ is the co-ordinate vector of x w.r.to β.

Problem 1:- Let $T : P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ defined by

$$T(f(x)) = f(x) + (x+1)f'(x).$$

Then find eigen vector of T.

Solⁿ:- We have, $\beta = \{1, x, x^2\}$ is the standard basis of $P_2(\mathbb{R})$.

$$\therefore T(1) = 1 = 1 \cdot 1 + x \cdot 0 + x^2 \cdot 0$$

$$T(x) = x + (x+1) \cdot 1 = 1 + 2x = 1 \cdot 1 + 2 \cdot x + 0 \cdot x^2$$

$$T(x^2) = x^2 + (x+1) \cdot 2x = 3x^2 + 2x = 0 \cdot 1 + 2 \cdot x + 3 \cdot x^2.$$

$$\therefore [T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

$\therefore$ The characteristic equⁿ is

$$|[T]_\beta - \lambda I_3| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\Rightarrow \lambda = 1, 2, 3$$

Since, all eigen values are distinct then T is diagonalis

We have to find eigen vector corresponding to $\lambda = 1$

$$\therefore \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_2 \\ x_2 + 2x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_1 = a, x_2 = 0, x_3 = 0, \; a \in F.$$

$\therefore$ Eigen vector of $[T]_\beta = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\therefore$ Eigen vector of T is $\varphi_\beta^{-1}\left\{a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$

$$= a\, \varphi_\beta^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= a \cdot 1$$

$\therefore$ The eigen vector of T w.r.to $\lambda = 1$ is 1.

The eigen vector corresponding $\lambda = 2$ is given by

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_1 = x_2, \; x_3 = 0.$$

Let $x_1 = a = x_2, \; x_3 = 0$.

$\therefore$ Eigen vector of $[T]_\beta$ w.r.to $\lambda = 2$ is $a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

$$\therefore a\,\varphi_\beta^{-1}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = a\{1\cdot 1 + 1\cdot x\} = a(1+x), \; a \in F.$$

$\therefore$ The eigen vector corresponding to $\lambda = 3$ is given by

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -2x_1 + x_2 = 0, \quad -x_2 + 2x_3 = 0$

Let $x_2 = a$, then $x_1 = \frac{a}{2}$, $x_3 = \frac{a}{2}$

$\therefore$ Eigen vector of $[T]_\beta$ w.r.to $\lambda = 3$ is $a\begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}$

lisable

$\therefore a\,\varphi_\beta^{-1}\begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix} = a\left\{\frac{1}{2} + x + \frac{1}{2}x^2\right\} = \frac{a}{2}(1 + 2x + x^2)$

$, a \in F$.

$\therefore$ For the basis $\beta' = \{1, 1+x, 1+2x+x^2\}$, the matrix $[T]_{\beta'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Jordan Canonical form :-

Definition :- The matrix of the form

$$J = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

is called Jordan block matrix belonging to λ. In this matrix A, λ's are on the diagonal and 1's are on the superdiagonal and other elements are zero.