Complex Analysis

Semester- 6th.

Paper: core T13

Course: Mathematics (H).

Chapter-1: Limit, Continuity & differentiability

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Part-1

Since $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$.

If for given $\epsilon > 0$, $\exists$ $\delta = \delta(\epsilon, z_0) > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

Let
$$\eta(z) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \quad \text{for } 0 < |z - z_0| < \delta$$
$$= 0 \qquad \text{for } z = z_0.$$

We have $\lim_{z \to z_0} \eta(z) = 0 = \eta(z_0)$. Therefore, η is continuous at z_0.

So, $f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\eta(z)$ for $|z - z_0| < \delta$ — (1)

Proposition: Let $D \subseteq \mathbb{C}$ be open, $f : D \to \mathbb{C}$ and $z_0 \in D$. Then $f'(z_0)$ exists iff $\exists$ a function $\eta : D \to \mathbb{C}$ which is continuous at z_0 and satisfies (1) for all $z \in D$. Equivalently, f is differentiable at z_0 iff

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + E(z) \quad \text{where } E \text{ is a}$$

function defined in a neighbourhood of z_0 s.t.
$$\lim_{z \to z_0} [E(z)] = 0.$$

Ex: Let $f(z) = z^2$. Let z_0 be any arbitrary point. Then P.T $f'(z_0) = 2z_0$

Ans: $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$

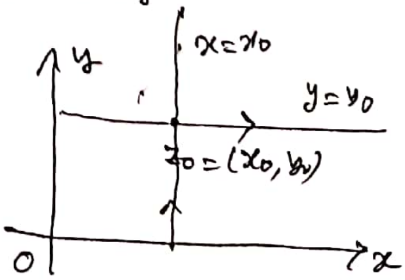
$$= \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0}$$

$$= \lim_{z \to z_0} (z + z_0) = 2z_0.$$

Ex: Let $f(z) = |z|^2$. Show that the derivative of $f(z)$ exists only at the origin.

Ans: Let $z_0 = x_0 + iy_0$ be a fixed point and $z = x + iy$.

Then $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{|z|^2 - |z_0|^2}{z - z_0}$. (1)



Let us approach z_0 along the line parallel to x-axis then
$$z = x + iy_0$$
$$z - z_0 = (x + iy_0) - (x_0 + iy_0) = (x - x_0)$$

So, $z \to z_0 \Rightarrow x \to x_0$. The above limit (1) becomes

$$= \lim_{x \to x_0} \frac{(x^2+y_0^2) - (x_0^2+y_0^2)}{x-x_0}$$

$$= \lim_{x\to x_0} \frac{x^2-x_0^2}{x-x_0} = \lim_{x\to x_0} (x+x_0) = 2x_0. \quad \text{———} (2)$$

Also we approach z_0 along the line parallel to y-axis, we have

$z = x_0 + iy$.

$z - z_0 = x_0+iy - (x_0+iy_0) = i(y-y_0)$.

So, $z \to z_0 \Rightarrow y \to y_0$.

Therefore, the limit (1) becomes

$$\lim_{y\to y_0} \frac{x_0^2+y^2-(x_0^2+y_0^2)}{i(y-y_0)}$$

$$= \lim_{y\to y_0} \frac{y^2-y_0^2}{i(y-y_0)} = \lim_{y\to y_0} \frac{-i^2(y_0+y)(y-y_0)}{i(y-y_0)}$$

$$= -i\,2y_0 \quad \text{———} (3)$$

From (2) and (3) it follows that $f(z) = |z|^2$ is not differentiable at a point $z_0 = x_0+iy_0$ when atleast one of x_0 and y_0 are non-zero.

<u>2nd Part</u>: When $x_0 = 0$ and $y_0 = 0$ then $z_0 = 0$, we have from (1)

$$\lim_{z\to 0} \frac{|z|^2 - 0}{z} = \lim_{z\to 0} \frac{z\bar{z}}{z} = \lim_{z\to 0} \bar{z} = 0.$$

Hence, the derivative of $f(z) = |z|^2$ exists only at the origin.

<u>Theorem</u>: If a function f is differentiable at z_0 then it must be continuous at z_0 but the converse is not true.

<u>Proof</u>: Let $f'(z_0)$ exist.

Then $\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z-z_0} = f'(z_0)$.

Now, $f(z) - f(z_0) = \frac{f(z)-f(z_0)}{z-z_0} \times (z-z_0)$

$$\lim_{z\to z_0} [f(z) - f(z_0)] = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \times \lim_{z\to z_0} (z-z_0)$$

$$= f'(z_0) \times 0 = 0$$

$\Rightarrow \lim_{z\to z_0} f(z) = f(z_0)$

$\Rightarrow f$ is continuous at z_0.

But the converse of the above theorem is not true.

Let $f(z) = |z|$

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$$

$$= \lim_{z\to 0} \frac{|z|}{z} \quad (1)$$

Let us approach origin along the real axis i.e. $y=0$.

$\therefore$ The above limit becomes

$$\lim_{x\to 0} \frac{\sqrt{x^2}}{x} = 1. \quad (2)$$

Let us approach origin along imaginary axis i.e. $x=0$.

The above limit becomes

$$\lim_{y\to 0} \frac{y}{iy} = \frac{1}{i} \quad (3)$$

From (2) & (3), it follows that $f(z)=|z|$ is not differentiable at $z=0$,

but $\lim_{z\to 0} f(z) = f(0) = 0$.

So, $f(z)$ is continuous at $z=0$.

Hence the result.

<u>Theorem</u>: (1) Let c be a complex constant and f be differentiable at a point z, then $\frac{d}{dz}[cf(z)] = cf'(z)$.

(2) If n be positive integer then $\frac{d}{dz}(z^n) = nz^{n-1}$.

This result is also true for n being negative integer, provided $z\neq 0$.

(3) If f and g be two functions differentiable at a point z, then (i) $\frac{d}{dz}[(f+g)z] = f'(z) \pm g'(z)$

(ii) $\frac{d}{dz}[(f\cdot g)z] = f'(z)g(z) + g'(z)f(z)$.

(iii) $\frac{d}{dz}[(f/g)z] = \frac{f'(z)g(z) - fg'(z)}{[g(z)]^2}$, $g(z)\neq 0$.

<u>Proof</u>: (2) Let n be positive integer, $f(z) = z^n$,

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z\to 0} \frac{(z+\Delta z)^n - z^n}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{z^n + {}^nc_1(\Delta z)^1 + {}^nc_2(\Delta z)^2 + \cdots + (\Delta z)^n - z^n}{\Delta z}$$

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$$= nz^{n-1} + \lim_{\Delta z \to 0} (\text{terms containing } \Delta z)$$

$$= nz^{n-1}.$$

$\boxed{\text{Ex}}$ (a) Show that $f'(z)$ does not exist at any point z when (i) $f(z) = \bar{z}$ (ii) $f(z) = \text{Re } z$ (iii) $f(z) = \text{Im } z$

(b) Let $f(z) = \dfrac{\bar{z}^2}{z}$, $z \neq 0$
$= 0$, $z = 0$. Show that $f'(0)$ does not exist.

(c) Let $f(z_0) = g(z_0) = 0$ and $f'(z_0)$ and $g'(z_0)$ both exist where $g'(z_0) \neq 0$.
Then show that, $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{f'(z_0)}{g'(z_0)}$.

Ans: (a) (i) Given that $f(z) = \bar{z}$. By defn, $f'(z) = \lim_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$.

Then $$f'(z) = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$
$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Let us approach origin along real axis, then $\Delta y = 0$.

So, $f'(z) = \lim_{\Delta x \to 0} \dfrac{\Delta x}{\Delta x} = 1.$ — (1)

Let us approach origin along the imaginary axis then $\Delta x = 0$.

So, $f'(z) = \lim_{\Delta y \to 0} \dfrac{-i\Delta y}{i\Delta y} = -1.$ — (2)

From (1) and (2), it follows that $f'(z)$ does not exist.

(ii) Given that $f(z) = \text{Re } z = x$ where $z = x + iy$.

$\therefore f(z + \Delta z) = \text{Re}(x + iy + \Delta x + i\Delta y) = x + \Delta x.$

Then $f'(z) = \lim_{(\Delta x, \Delta y) \to (0,0)} \dfrac{x + \Delta x - x}{\Delta x + i\Delta y}$ $\quad [\because \Delta z = \Delta x + i\Delta y]$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x}{\Delta x + i\Delta y}.$$

Let us approach origin along real axis, then $\Delta y = 0$.

We have $f'(z) = \lim_{\Delta x \to 0} \dfrac{\Delta x}{\Delta x} = 1.$

Let us approach origin along the imaginary axis, then $\Delta x = 0$.

We have $f'(z) = \lim_{\Delta y \to 0} \dfrac{0}{i\Delta y} = 0.$

Hence, from above, we see that $f'(z)$ does not exist.

(iii) Try yourself

(d) Given $f(z) = \dfrac{\bar{z}^2}{z}$

$$\therefore f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$$

$$= \lim_{z\to 0} \frac{\bar{z}^2}{z^2} = \lim_{z\to 0} \left(\frac{\bar{z}}{z}\right)^2$$

$$= \lim_{(x,y)\to(0,0)} \left(\frac{x-iy}{x+iy}\right)^2 .$$

Let us approach origin along the line $y = mx$ we have

$$f'(0) = \lim_{x\to 0} \frac{(x-imx)^2}{(x+imx)^2}$$

$$= \frac{(1-im)^2}{(1+im)^2} = \frac{(1-im)^4}{(1+m^2)^2}$$ which is different for different for different values of m.

So $f'(0)$ does not exist.

(e) $$\lim_{z\to z_0} \frac{f(z)}{g(z)}$$

$$= \lim_{z\to z_0} \frac{f(z)-f(z_0)}{g(z)-g(z_0)} \quad [\because f(z_0) = g(z_0) = 0]$$

$$= \lim_{z\to z_0} \frac{\frac{f(z)-f(z_0)}{z-z_0}}{\frac{g(z)-g(z_0)}{z-z_0}} \quad [z \neq z_0]$$

$$= \frac{f'(z_0)}{g'(z_0)}, \quad g'(z_0) \neq 0$$

$\boxed{Ex}$: Show that a polynomial $p(z)$ of degree $n\ (\geq 1)$ where $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0\ (a_n \neq 0)$ is differentiable everywhere with $p'(z) = n a_n z^{n-1} + \cdots + 2a_2 z + a_1$ and hence show that $\dfrac{p^n(0)}{\lfloor n} = a_n$.

Ans: We know that $\frac{d}{dz}(z^n) = n z^{n-1}$ (To be proved).

So, $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0 \quad (a_n \neq 0)$

$\therefore p'(z) = n a_n z^{n-1} + \cdots + a_1$

Differenting n times we have,

$p^n(z) = n(n-1)(n-2)\cdots 1\, a_n$

$\therefore p^n(0) = \lfloor n\, a_n \Rightarrow a_n = \dfrac{p^n(0)}{\lfloor n}$ (Proved)

$\boxed{\text{Ex}}$: Let $f(z) = \frac{|z|}{Re(z)}$ if $Re(z) \neq 0$

$= 0, \quad Re(z) = 0$.

Show that f is not continuous at $z=0$.

Ans : Let $z = x+iy$, $|z| = \sqrt{x^2+y^2}$

Then $f(z) = \frac{\sqrt{x^2+y^2}}{x}, \quad x \neq 0$

$= 0 \quad x = 0$.

We approach origin along the real axis ie $y=0$.

we have $\lim\limits_{x\to 0} \frac{x}{x} = 1$.

We approach origin along the imaginary axis ie along $x=0$

Then $\lim\limits_{y\to 0} \frac{y}{x} = 0$.

$\therefore \lim\limits_{z\to 0} f(z)$ does not exist.

Hence f is not continuous at $z=0$.

Complex Analysis

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Chapter 2: Analytic Function.

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Part-2.

Analytic Function and Cauchy Riemann's Equation:

Analytic Function:

A function f is said to be analytic or holomorphic at a point $a \in \mathbb{C}$ if it is differentiable at every point of some neighbourhood of a.

f is analytic on an arbitrary set S if it is differentiable at every point of some open set containing S.

A point at which f is analytic is called an ordinary point of f and a point where f is not analytic is called singular point.

Note: $z=0$ is a singularity of $f(z)=|z|^2$ even though f is differentiable at $z=0$. As $f(z)=|z|^2$ is nowhere differentiable except at $z=0$ so, f is not analytic at $z=0$.

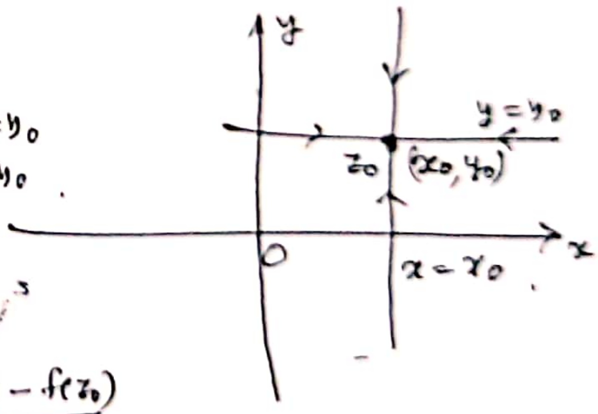
Example: Let $f(z)=\frac{1}{z}$. Here the point $z=0$ is a singular point of the function f as $f'(z)$ exists $\forall z \in (-\delta, \delta)$ except at $z=0$, $\delta>0$.

Theorem: A necessary condition that the function $f(z)=u(x,y)+iv(x,y)$ is differentiable at a point $z_0=x_0+iy_0$ is that the partial derivatives u_x, u_y, v_x, v_y exists and $u_x=v_y$ and $u_y=-v_x$ at (x_0,y_0).

[Necessary condition for a function to be differentiable].

Proof: Let $z_0=x_0+iy_0$ be any fixed point in D. Since $f'(z_0)$ exists,

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} \text{ exists.}$$



Let us approach z_0 along the line parallel to real axis $y=y_0$

$z=x+iy_0$

then $\Delta z = z-z_0$

$= x+iy_0-(x_0+iy_0)$

$=(x-x_0)=\Delta x$.

$\Delta z\to 0$ as $z\to z_0$

Thus, we have $f'(z_0)=\lim_{\Delta z\to 0}\frac{f(z)-f(z_0)}{z-z_0}$

$$=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

$$=\lim_{\Delta x\to 0}\left[\frac{u(x_0+\Delta x,y_0)-u(x_0,y_0)}{\Delta x}+i\,\frac{v(x_0+\Delta x,y_0)-v(x_0,y_0)}{\Delta x}\right]$$

Since the limit on L.H.S exists. So, the individual limits of $u(x,y)$ and $v(x,y)$ exists. So, $u_x(x_0,y_0)$ and $v_x(x_0,y_0)$ both exist.

So, $f'(z_0) = u_x(x_0,y_0) + i v_x(x_0,y_0)$ —— (1)

We approach z_0 along the line $x = x_0$ parallel to imaginary axis,

Then, $\Delta z = z - z_0 = (x_0 + iy) - (x_0 + iy_0) = i(y - y_0) = i\Delta y$.

So, $$f'(z_0) = \lim_{\Delta y \to 0}\left[\frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{i\Delta y} + i\,\frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{i\Delta y}\right]$$

$$= -i u_y(x_0,y_0) + v_y(x_0,y_0) \quad \text{—— (2)}$$

[$\because$ Since $u_y(x_0,y_0)$ and $v_y(x_0,y_0)$ exists]

Since the derivative of f at z_0 is unique, it follows from (1) and (2) that $u_x(x_0,y_0) + i v_x(x_0,y_0) = -i u_y(x_0,y_0) + v_y(x_0,y_0)$

$\Rightarrow$ $u_x(x_0,y_0) = v_y(x_0,y_0)$ and $u_y(x_0,y_0) = -v_x(x_0,y_0)$.

ie $u_x(x,y) = v_y$ & $u_y = -v_x$ at (x_0,y_0).

These equations are known as Cauchy Riemann Equation (C-R equations).

Ex-1 Let $f(z) = |z|^2$. The Cauchy Riemann equations are satisfied at the point $z=0$ but $f(z) = |z|^2$ is not differentiable at any point $z \neq 0$.

Ans: Given that $f(z) = |z|^2$
$= x^2 + y^2$
$= u(x,y) + i v(x,y)$.

So, $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$.

Therefore, $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$.

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $u_y = -v_x$ at $(0,0)$.

So the C-R equations are satisfied at $(0,0)$.

2nd part is proved previously.

Theorem: (Sufficient condition for differentiability)

The sufficient conditions for a single valued continuous function $f(z) = u(x,y) + iv(x,y)$ to be analytic in a domain D are

(i) The partial derivatives u_x, v_x, u_y, v_y exist and are continuous

(ii) They satisfy the C-R equations at each point of D i.e,

$u_x = v_y$ and $u_y = -v_x$.

Proof: Since $u(x,y)$, $v(x,y)$ and its partial derivatives of the 1st order are continuous in a domain D and satisfy C-R equations at each point of D. We have to prove that $f(z)$ is analytic in D.

Let $(x+\delta x, y+\delta y)$ be any point in some n.bd. of (x,y).

Now, $$u(x+\delta x, y+\delta y) - u(x,y) = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \epsilon_1 \delta x + \epsilon_2 \delta y$$

where $\epsilon_1, \epsilon_2 \to 0$ as $(\delta x, \delta y) \to (0,0)$ — (1)

Similarly, $$v(x+\delta x, y+\delta y) - v(x,y) = \frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y + \epsilon_3 \delta x + \epsilon_4 \delta y$$

where $\epsilon_3, \epsilon_4 \to 0$ as $(\delta x, \delta y) \to (0,0)$. — (2)

Let $\delta z = \delta x + i\delta y$.

Then $f(z+\delta z) - f(z)$

$$= [u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)] - [u(x,y) + iv(x,y)]$$

$$= [u(x+\delta x, y+\delta y) - u(x,y)] + i[v(x+\delta x, y+\delta y) - v(x,y)]$$

$$= \left[\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \epsilon_1 \delta x + \epsilon_2 \delta y\right] + i\left[\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y + \epsilon_3 \delta x + \epsilon_4 \delta y\right]$$

$$= \left[\frac{\partial u}{\partial x}\delta x - \frac{\partial v}{\partial x}\delta y + \epsilon_1 \delta x + \epsilon_2 \delta y\right] + i\left[\frac{\partial v}{\partial x}\delta x + \frac{\partial u}{\partial x}\delta y + \epsilon_3 \delta x + \epsilon_4 \delta y\right]$$

$$= \frac{\partial u}{\partial x}(\delta x + i\delta y) + i\frac{\partial v}{\partial x}(\delta x + i\delta y) + (\epsilon_1 + i\epsilon_3)\delta x + (\epsilon_2 + i\epsilon_4)\delta y$$

$[\because u_x = v_y \ \& \ u_y = -v_x]$

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = u_x + iv_x + (\epsilon_1 + i\epsilon_3)\frac{\delta x}{\delta x + i\delta y} + \frac{(\epsilon_2 + i\epsilon_4)\delta y}{\delta x + i\delta y}$$

$$\Rightarrow f'(z) = u_x + iv_x$$ $\left[\because \left|\frac{\delta x}{\delta x + i\delta y}\right| \le 1, \left|\frac{\delta y}{\delta x + i\delta y}\right| \le 1 \text{ and } \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \to 0 \text{ as } (\delta x, \delta y) \to (0,0)\right]$

Hence $f'(z)$ exists.

Theorem: A real valued function of a Complex variable either has derivative zero or the derivative does not exist.

Proof: Let f is a real function of complex variable z whose derivative exists at z_0,

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \quad \text{where } h = h_1 + ih_2,$$

If we take the limit $h \to 0$ along the real axis i.e. $h = h_1$, we have

$$f'(z_0) = \lim_{h_1 \to 0} \frac{f(z_0+h_1) - f(z_0)}{h_1} = \text{a real number} \quad \text{--- (1)}$$

[Since f is real valued function of complex variable].

Also, we take limit $h \to 0$ along imaginary axis

i.e. $h = ih_2$

$$\therefore f'(z_0) = \lim_{h_2 \to 0} \frac{f(z_0+ih_2) - f(z_0)}{ih_2}$$

$$= \frac{1}{i} \times \text{real value} \quad \text{--- (2)}.$$

$=$ Purely imaginary number.

From (1) and (2), we have $f'(z_0) = 0$

Hence, either derivative zero or does not exist.

$\boxed{\text{Ex}}$: **Theorem:** Let f be analytic in a region G. Then

(i) If $f'(z) = 0$ on G then f is constant on G

(ii) If any one of $\text{Re} f$, $\text{Im} f$, $|f|$ is constant on G then f is constant on G.

Proof: (i) Since f be analytic in a region G.

$$\therefore f'(z) = u_x + iv_x$$
$$= v_y - iu_y \quad [u_x = v_y \ \& \ u_y = -v_x]$$

$$\therefore f'(z) = 0 \Rightarrow u_x = v_x = u_y = v_y = 0, \ \forall z \in G.$$

Since $u_x = u_y = 0 \Rightarrow u$ is not a function of x and y.

$\therefore u$ is constant on G.

Similarly, $v_x = v_y = 0 \Rightarrow v$ is constant on G.

Hence, $f(z)$ is constant on G.

(ii) Let $|f(z)| = K \;\forall\; z \in G$. $K \neq 0$.

$\therefore$ $u^2 + v^2 = K^2$ — ①

Differentiating ① w.r.to x and y we get

$u u_x + v v_x = 0$ .... ②

& $u u_y + v v_y = 0$ .... ③

② and ③ becomes $u u_x - v u_y = 0$
and $u u_y + v u_x = 0$ [using C-R equation $u_x = v_y$, $u_y = -v_x$]

Squaring and adding above we have

$$(u_x^2 + u_y^2)(u^2 + v^2) = 0$$

$$\Rightarrow K^2 (u_x^2 + u_y^2) = 0$$

$$\Rightarrow K^2 |f'(z)|^2 = 0 \qquad \left[\begin{aligned} f'(z) &= u_x + i v_x \\ &= u_x - i u_y \end{aligned} \right].$$

Since $K \neq 0$, $f'(z) = 0$ in $G \Rightarrow f$ is constant in G

Also, $u = \text{Re} f$ is constant. Then $u_x = u_y = 0$.

$\Rightarrow v_x = v_y = 0$ [By C-R equation]

$\therefore$ $f'(z) = 0$ in G.

Hence f is constant.

Also, $v = \text{Im} f$ is constant. So, $v_x = v_y = 0$

$\Rightarrow u_x = u_y = 0$ [By C-R equations]

$\therefore$ $f'(z) = 0$ in G.

Hence f is constant.

$\boxed{\text{Ex}}$ (a) Examine the nature of the function $f(z) = \dfrac{x^2 y^5 (x+iy)}{x^4 + y^{10}}$, $z \neq 0$
$= 0$, $z = 0$.
in the region including origin.

Ans: Given that $f(z) = \dfrac{x^2 y^5 (x+iy)}{x^4 + y^{10}} = u + iv$.

$\therefore$ $u(x,y) = \dfrac{x^3 y^5}{x^4 + y^{10}}$, $v(x,y) = \dfrac{x^2 y^6}{x^4 + y^{10}}$.

At origin, $u_x = \lim\limits_{h \to 0} \dfrac{u(h,0) - u(0,0)}{h}$

$$= \lim_{h \to 0} \frac{0-0}{0} = 0.$$

$$u_y(0,0) = \lim_{k\to 0} \frac{u(0,k) - u(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{0-0}{k} = 0$$

$$v_x(0,0) = \lim_{h\to 0} \frac{v(h,0) - v(0,0)}{h} = 0 = v_y(0,0).$$

Hence, Cauchy Riemann equations are satisfied at the origin.

$$\text{Now, } f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{z\to 0} \frac{f(z)}{z}$$

$$= \lim_{(x,y)\to(0,0)} \left[\frac{x^2 y^5 (x+iy)}{x^4 + y^{10}} \right] \cdot \frac{1}{x+iy}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2 y^5}{x^4 + y^{10}}.$$

We approach $z\to 0$ along the line $y = mx$ we have

$$f'(0) = \lim_{x\to 0} \frac{m^5 x^2 x^5}{x^4 + m^{10} x^{10}}$$

$$= 0$$

We approach $z\to 0$ along the curve $y^5 = mx^2$ we have

$$f'(0) = \lim_{x\to 0} \frac{x^2 m x^2}{x^4 + m^2 x^4}$$

$$= \lim_{x\to 0} \frac{m}{1+m^2}$$ which is different for different values of m.

$$= \frac{m}{1+m^2}$$

$\therefore$ So $f'(0)$ does not exist.

$\boxed{\text{Ex}}$: If $z = x+iy$ and $f(z) = \dfrac{\bar{z}^2}{z}$ for $z \neq 0$

$= 0 \quad z = 0.$

Show that C-R equations are satisfied at $z=0$ but $f'(0)$ does not exist.

Ans: Given that $f(z) = \dfrac{\bar{z}^2}{z}.$

$$\therefore f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{z\to 0} \frac{\bar{z}^2}{z^2} = \lim_{z\to 0} \left(\frac{\bar{z}}{z}\right)^2$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x-iy)^2}{(x+iy)^2} = \lim_{(x,y)\to(0,0)} \left(\frac{x-iy}{x+iy}\right)^2$$

Let us approach $z \to 0$ along the line $y = mx$ we have

$$f'(0) = \lim_{x \to 0} \frac{(x - imx)^2}{(x + imx)^2}$$

$$= \frac{(1 - im)^2}{(1 + im)^2}$$

$$= \frac{(1 - im)^4}{(1 + m^2)^2}$$ which is different for different values of m.

$\therefore$ $f(z)$ is not differentiable at $z = 0$.

Also, $f(z) = \frac{\bar{z}^2}{z} = \frac{(x - iy)^2}{x + iy} = \frac{(x^2 - y^2 - 2ixy)(x - iy)}{(x + iy)(x - iy)}$

$$= \frac{(x^3 - 3xy^2) - i(3x^2y - y^3)}{x^2 + y^2}$$

$= u + iv$

where $u = \frac{x^3 - 3xy^2}{x^2 + y^2}$, $v(x, y) = \frac{y^3 - 3x^2y}{x^2 + y^2}$.

Now, $u_x(0,0) = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h \to 0} \frac{h - 0}{h} = 1$.

$u_y(0,0) = \lim_{k \to 0} \frac{u(0,k) - u(0,0)}{k} = \lim_{k \to 0} \frac{0 - 0}{k} = 0$.

$v_x(0,0) = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0$.

$v_y(0,0) = \lim_{k \to 0} \frac{v(0,k) - v(0,0)}{k} = \lim_{k \to 0} \frac{k - 0}{k} = 1$

$\therefore$ $u_x(0,0) = v_y(0,0)$ and $u_y(0,0) = -v_x(0,0)$.

Hence, $f(z)$ satisfies C-R equations at $z = 0$.

Ex: Let $f(z) = \frac{x^3y(y - ix)}{x^6 + y^2}$, $z \neq 0$

$= 0$, $z = 0$.

Show that $\frac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

Ans: Let us approach $z \to 0$ along the radius vector $y = mx$ we have

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{x^3y(y - ix)}{(x^6 + y^2)(x + iy)}$$

$$= \lim_{x \to 0} \frac{x^3 \cdot mx(mx - ix)}{(x^6 + m^2x^2)(x + imx)} = 0$$

Let us approach $z \to 0$ along the curve $y = x^3$ we have

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{(x,y)\to(0,0)} \frac{x^3 y\,(y - ix)}{x^6 + y^2} \cdot \frac{1}{x+iy}$$

$$= \lim_{x\to 0} \frac{x^3 \cdot x^3 (x^3 - ix)}{(x^6 + x^6)(x + ix^3)}$$

$$= -\frac{i}{2}$$

Hence the result.

$\boxed{\text{Ex}}$: Show that the function $f(z) = \sqrt{|xy|}$ is not differentiable at the origin although C-R equations are satisfied at that point.

Ans: Given function $f(z) = \sqrt{|xy|}$.

Here $u(x,y) = \sqrt{|xy|}$ & $v(x,y) = 0$.

Now, $u_x(0,0) = \lim_{h\to 0} \frac{u(h,0) - u(0,0)}{h} = 0$,

$u_y(0,0) = \lim_{k\to 0} \frac{u(0,k) - u(0,0)}{k} = 0$

$v_x(0,0) = \lim_{h\to 0} \frac{v(h,0) - v(0,0)}{h} = 0$, $v_y(0,0) = \lim_{k\to 0} \frac{v(0,k) - v(0,0)}{k} = 0$

Hence C-R equations are satisfied at the origin.

Also, $f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0}$

$$= \lim_{(x,y)\to(0,0)} \frac{\sqrt{|xy|}}{x+iy}.$$

Let us approach $z \to 0$ along the line $y = mx$ we have

$$= \lim_{x\to 0} \frac{\sqrt{|mx^2|}}{x + imx}$$

$$= \frac{\sqrt{|m|}}{1+im}.$$

which is different for different values of m.

Therefore, $f(z)$ is not analytic at the origin although C-R equations are satisfied at $z=0$

$\boxed{\text{Ex}}$: Let $f(z) = \frac{x^3 - y^3}{x^2+y^2} + i\,\frac{(x^3+y^3)}{x^2+y^2}$, $z \neq 0$.

$= 0 \qquad z = 0$

Show that the function f satisfies the C-R equations at the origin but f is not differentiable at $z=0$.

Ans: Given that $f(z) = \frac{x^3-y^3}{x^2+y^2} + i\,\frac{(x^3+y^3)}{x^2+y^2}$

$= u+iv$

where $u = \frac{x^3-y^3}{x^2+y^2}$ and $v = \frac{x^3+y^3}{x^2+y^2}$

$$u_x(0,0) = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1$$

$$u_y(0,0) = \lim_{k\to 0} \frac{u(0,k)-u(0,0)}{k} = \lim_{k\to 0} \frac{-k-0}{k} = -1.$$

$$v_x(0,0) = \lim_{h\to 0} \frac{v(h,0)-v(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1.$$

$$v_y(0,0) = \lim_{k\to 0} \frac{v(0,k)-v(0,0)}{k} = \lim_{k\to 0} \frac{k}{k} = 1.$$

So, $u_x(0,0) = v_y(0,0)$ and $u_y(0,0) = -v_x(0,0)$.

Hence, the Cauchy Riemann equations are satisfied at the origin.

Now, $f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$

$$= \lim_{z\to 0} \left(\frac{x^3-y^3}{x^2+y^2} + i\,\frac{x^3+y^3}{x^2+y^2}\right)\Big/(x+iy).$$

We approach $z\to 0$ along the line $y=0$ then

$$f'(0) = \lim_{x\to 0} \frac{(x+ix)}{x} = 1+i \quad \text{—— (1)}$$

We approach $z\to 0$ along the line $y=x$ we have

$$f'(0) = \lim_{x\to 0} \frac{ix}{x+ix} = \frac{i}{1+i} = \frac{1+i}{2}. \quad \text{—— (2)}$$

From (1) and (2), it follows that $f'(0)$ does not exist.

Harmonic Functions :

Def^n : Any function u of x and y which possesses continuous partial derivatives of the 1st and 2nd orders and satisfies Laplace's equation i.e. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, is called the harmonic functions.

Theorem: If $f(z) = u+iv$ is an analytic function then u and v are harmonic functions.

Proof: Let $f(z) = u+iv$ be an analytic function.

So, C-R equation are satisfied.

i.e. $u_x = v_y$ and $u_y = -v_x$

Since u & v are continuous and derivatives of u and v of all order exist and continuous functions of x and y.

So, $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$ and $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Now, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Similarly, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$.

Hence, both u and v satisfy Laplace's equation. Therefore, both u & v are harmonic functions.

$\boxed{\text{Ex}}$: Show that an analytic function with constant modulus is constant.

Proof is done previously.

$\boxed{\text{Ex}}$: Show that the function $f(z) = e^{-z^{-4}}$ $(z \neq 0)$ and $f(0) = 0$ is not differentiable at $z=0$ although C-R equations are satisfied at $z=0$.

Ans: Given function is $f(z) = e^{-z^{-4}}$

$$= e^{-\frac{1}{z^4}}$$

$$= e^{-\frac{1}{(x+iy)^4}}$$

$$= e^{-\frac{(x-iy)^4}{(x^2+y^2)^4}}$$

$$= e^{-\frac{1}{(x^2+y^2)^4}\left[x^4+y^4-6x^2y^2-4ixy(x^2-y^2)\right]}$$

$$= u+iv \text{ (say)}$$

where

$$u = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}} \cos\left\{\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right\}$$

$$v = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}} \sin\left\{\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right\}$$

Now, $u_x(0,0) = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h} = \lim_{h\to 0} \frac{e^{-h^{-4}}}{h} = 0$

$u_y(0,0) = \lim_{k\to 0} \frac{u(0,k)-u(0,0)}{k} = \lim_{k\to 0} \frac{e^{-k^{-4}}}{k} = 0$.

$v_x(0,0) = \lim_{h\to 0} \frac{v(h,0)-v(0,0)}{h} = 0$.

$$v_y(0,0) = \lim_{k\to 0} \frac{v(0,k) - v(0,0)}{k} = \lim_{k\to 0} \frac{0}{k} = 0.$$

$\therefore u_x(0,0) = v_y(0,0)$ and $u_y(0,0) = -v_y(0,0)$.

Hence Cauchy Riemann's equation are satisfied at $z=0$

$$\text{But } f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z}$$
$$= \lim_{z\to 0} \frac{e^{-z^{-4}}}{z}$$

Let us approach $z \to 0$ along $z = re^{i\theta}$, θ is any real no.

$$\therefore f'(0) = \lim_{r\to 0} \frac{e^{-(re^{i\theta})^{-4}}}{re^{i\theta}} \quad \text{does not exist.}$$

$\therefore$ $f(z)$ may not analytic at $z=0$

$\boxed{\text{Ex}}$: If $f(z)$ is an analytic function of z Prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[\operatorname{Re} f(z)]^2 = 2\,|f'(z)|^2$$

Proof: Let $f(z) = u + iv$ so $\operatorname{Re}(f(z)) = u$

$$\frac{\partial}{\partial x}(u^2) = 2u\frac{\partial u}{\partial x}$$

$$\frac{\partial^2}{\partial x^2}(u^2) = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2u\frac{\partial^2 u}{\partial x^2} \quad \text{(1)}$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2) = 2\left(\frac{\partial u}{\partial y}\right)^2 + 2u\frac{\partial^2 u}{\partial y^2}$ — (2)

Hence, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2) = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + 2u\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$

$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$ [u is harmonic function as $f(z)$ is analytic].

$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$ [$\because u_y = -v_x$]

$= 2\left[|f'(z)|\right]^2$ [$\because f'(z) = u_x + iv_x$]

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[\operatorname{Re} f(z)]^2 = 2\,|f'(z)|^2.$$

$\boxed{\text{Ex}}$: Show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z\,\partial \bar z}$

Ans: We have $z = x+iy$ and $\bar z = x - iy$

$$x = \tfrac{1}{2}(z+\bar z) \quad \text{and} \quad y = \tfrac{1}{2i}(z - \bar z)$$
$$= -\tfrac{i}{2}(z-\bar z)$$

Therefore, $\frac{\partial x}{\partial z} = \frac{1}{2}$, $\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$, $\frac{\partial y}{\partial z} = -\frac{i}{2}$, $\frac{\partial y}{\partial \bar{z}} = \frac{i}{2}$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Hence, $$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}.$$

Ex: If $f(z)$ is an analytic function of z in any domain, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2 |f(z)|^{p-2} |f'(z)|^2$.

Ans: We have $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

So, $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = 4\frac{\partial^2}{\partial z \partial \bar{z}}|f(z)|^p$$

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}}\left\{f(z) f(\bar{z})\right\}^{p/2}$$

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}}\left[\{f(z)\}^{p/2}\{f(\bar{z})\}^{p/2}\right]$$

$$= 4\frac{\partial}{\partial z}\left[[f(z)]^{p/2}\frac{p}{2}\{f(\bar{z})\}^{p/2-1}.f'(\bar{z})\right]$$

$$= 4\left[\frac{p}{2}\{f(z)\}^{p/2-1}.f'(z)\,\frac{p}{2}\{f(\bar{z})\}^{p/2-1}f'(\bar{z})\right]$$

$$= p^2\{f(z) f(\bar{z})\}^{p/2-1}.f'(z)\, f'(\bar{z})$$

$$= p^2\left[|f(z)|^2\right]^{p/2-1}.|f'(z)|^2$$

$$= p^2 |f(z)|^{p-2}|f'(z)|^2$$

How to determine the conjugate function:

If $f(z) = u + iv$ is an analytic function then both u & v are conjugate functions. If any one of these say $u(x,y)$ is given, then we have to determine the other $v(x,y)$.

Since v is a function of x and y,

$$\therefore dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \left[\text{By C-R equations } \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right] \quad \text{---} (1)$$

This is of the form $M\,dx + N\,dy$.

Where $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$

$$\frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

Since u is a harmonic function, so it satisfy Laplace's equation.

i.e. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow -\frac{\partial M}{\partial y} = -\frac{\partial N}{\partial x} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

This shows that (1) satisfy the condition of exact differential equation.

So, the equation (1) can be integrated & then v is determined.

$\boxed{\text{Ex}}$: Show that the function $u(x,y) = \cos x \cosh y$ is harmonic and find its harmonic conjugate function v.

Ans: Given $u(x,y) = \cos x \cosh y$.

$$\frac{\partial u}{\partial x} = -\sin x \cosh y$$

$$\frac{\partial^2 u}{\partial x^2} = -\cos x \cosh y \qquad \frac{\partial u}{\partial y} = \cos x \sinh y$$

$$\frac{\partial^2 u}{\partial y^2} = \cos x \cosh y.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

So, u satisfy Laplace's equation.

Hence, u is a harmonic function.

2nd Part: If v be its conjugate harmonic function then the function $f = u + iv$ must be analytic and C-R equations must be satisfied by f.

So $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$.

$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$

$= -\cos x \sinh y\, dx + \sin x \cosh y\, dy$

$= -\, d(\sin x \sinh y)$

Int, $v = -\sin x \sinh y + c$, c is real constant.

$\boxed{Ex}$: Prove that the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ satisfies Laplace's equation and determine analytic function.

Ans: Here $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$, $\frac{\partial u}{\partial y} = -6xy - 6y$

$\frac{\partial^2 u}{\partial x^2} = 6x + 6$ $\qquad \frac{\partial^2 u}{\partial y^2} = -6x - 6.$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

u satisfy the Laplace's equation. Hence, u is harmonic function.

Now, $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ $\left[u_x = v_y \text{ and } u_y = -v_x\right]$

$= (6xy + 6y)\,dx + (3x^2 - 3y^2 + 6x)\,dy$

$= 6\,d(xy) + d(3x^2y) - 3y^2 dy$

Int, $v = 6xy + 3x^2y - y^3 + c.$

Now $f(z) = u + iv$

$= x^3 - 3xy^2 + 3x^2 - 3y^2 + 1 + i\,(6xy + 3x^2y - y^3 + c)$

$= (x+iy)^3 + 3(x+iy)^2 + 1 + ic$

$= z^3 + 3z^2 + k$, k being constant.

Milne's Thomson Method [To construct f(z)]

In this method $u(x,y)$ is given but $f(z)$ is determined without finding v

Let $z = x+iy$, $\bar{z} = x - iy$

$\therefore x = \frac{1}{2}(z+\bar{z})$ and $y = \frac{1}{2i}(z - \bar{z}).$

So, we have $f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + i v\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$

Put $z = \bar{z}$ we have, $x = z$, $y = 0$. Then

$f(z) = u(z,0) + i v(z,0)$

We have $f(z) = u + iv$. Therefore,

$$f'(z) = u_x + i v_x$$
$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad [\text{By C-R equations}]$$

If we take $\frac{\partial u}{\partial x} = \phi_1(x,y)$ and $\frac{\partial u}{\partial y} = \phi_2(x,y)$ we have,

$$f'(z) = \phi_1(x,y) - i\phi_2(x,y)$$
$$= \phi_1(z,0) - i\phi_2(z,0)$$

Integrating, $f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + c$

where c being an arbitrary constant.

Thus, the function $f(z)$ is constructed when $u(x,y)$ is given. Similarly, if $v(x,y)$ is given, it can be shown that

$$f(z) = \int [\psi_1(z,0) + i\psi_2(z,0)]\, dz + c.$$

where $\psi_1(x,y) = \frac{\partial v}{\partial y}$ and $\psi_2(x,y) = \frac{\partial v}{\partial x}$.

$\boxed{\text{Ex}}$: Prove that $u = e^x(x\cos y - y\sin y)$ satisfy Laplace's equation and find the corresponding analytic function $f(z) = u + iv$.

Ans: Given that $u = e^x(x\cos y - y\sin y)$.

$$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x(\cos y)$$
$$= e^x(x\cos y - y\sin y + \cos y) = \phi_1(x,y)$$

$$\frac{\partial u}{\partial y} = e^x(-x\sin y - y\cos y - \sin y) = \phi_2(x,y)$$

Hence, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Try yourself)

So, u satisfies Laplace's equation.

By Milne's method,

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$$
$$= e^z(z+1)$$

Int, $f(z) = \int e^z z\, dz + \int e^z dz + c$

$= e^z.z - e^z + e^z + c = ze^z + c$

c being constant.

$\boxed{Ex}$: If $u = x^3 - 3xy^2$ show that there exists a function $v(x,y)$ s.t $f(z) = u + iv$ is analytic in a finite region.

Ans: Given that $u(x,y) = x^3 - 3xy^2$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \phi_1(x,y) \qquad \frac{\partial u}{\partial y} = -6xy = \phi_2(x,y).$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

$\therefore$ u satisfies Laplace's equation. So it is harmonic function.

By Milne's method

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$$
$$= 3z^2 + 0$$

Integrating, $f(z) = z^3 + c$, c being arbitrary constant.

Here $f(z)$ is also analytic in any finite region.

$\boxed{Ex}$: Let $u(x,y) = e^x \cos y$. To determine $v(x,y)$ such that $f(z) = u + iv$ is analytic.

Ans: Try yourself.

$\boxed{Ex}$: (a) Determine $f(z) = u + iv$ by determining a harmonic conjugate of a given harmonic function $u(x,y) = y^3 - 3x^2 y$.

(b) Show that $u(x,y) = \sinh x \sin y$ is harmonic in some domain D and find harmonic conjugate $v(x,y)$.

Ans: Try your self.

$\boxed{Ex}$ (a) If $f(z)$ be analytic in a domain D then show that $f(z)$ must be constant in D if (i) $f(z)$ is real valued $\forall z \in D$. (ii) $\overline{f(z)}$ is analytic in D (iii) $\arg f(z) =$ constant.

(b) If $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$ then find analytic function $f(z) = u + iv$

(c) Show that $u(x,y) = \frac{1}{2}\log(x^2+y^2)$ is harmonic and find its conjugate harmonic

(d) If $f(z)$ is analytic function of $z = x+iy$ then $\frac{\partial f}{\partial \bar{z}} = 0$ $\left[\frac{\partial f}{\partial \bar{z}} = 0\right]$

Ans: (a) (i) Since $f(z)$ is analytic. So Cauchy-Riemann's equations hold.

So $u_x = v_y$ and $u_y = -v_x$.

Also $f(z) = u + iv$. and if $f(z)$ is real valued function $\forall z \in D$.

Then $v = 0$,

$$\therefore \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0,$$

$$\Rightarrow \frac{\partial u}{\partial y} = 0, \quad \Rightarrow \frac{\partial u}{\partial x} = 0$$

Thus, $\frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$.

Then $u(x,y)$ and $v(x,y)$ are constant.

$\therefore$ $f(z)$ is constant

(ii) Since $f(z) = u+iv$ and $f(z)$ is analytic. So, $u_x = v_y$ and $u_y = -v_x$ — (1)

Let $\bar{f}(z) = u-iv$ is analytic. Then Cauchy-Riemann's equations are satisfied.

So, $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\left(-\frac{\partial v}{\partial x}\right) = \frac{\partial v}{\partial x}$

Using (1), we have $\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial y}$

$\Rightarrow \frac{\partial v}{\partial y} = 0$ and $-\frac{\partial v}{\partial x} = \frac{\partial v}{\partial x} \Rightarrow \frac{\partial v}{\partial x} = 0$.

Also, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$ [using (1)].

So, u and v both are constant.

Hence, $f(z)$ is constant.

(iii) $\text{Arg } f(z) = \text{constant}$. Where $f(z) = u+iv$.

$\Rightarrow \tan^{-1}\left(\frac{v}{u}\right) = \text{constant} = k$ (say)

$\Rightarrow \frac{v}{u} = \tan k \Rightarrow v = u\tan k$,

$\Rightarrow v = cu$ — (1) [$c = \tan k$ is real constant]

Since $f(z)$ is analytic so C-R equations hold.

$\therefore$ $u_x = v_y$ and $u_y = -v_x$ — (2)

From (1) $v_x = cu_x$ and $v_y = cu_y$ — (3)

Thus $cu_x = -u_y$ and $cu_y = u_x$ [using (2) & (3)]

$\Rightarrow cu_x + u_y = 0$ — (4)

and $u_x - cu_y = 0$. — (5)

Sq and adding (4) and (5) we have $c^2(u_x^2+u_y^2) + u_x^2+u_y^2 = 0$

$\Rightarrow (c^2+1)(u_x^2+v_x^2) = 0$ [$\because u_y = -v_x$]

$\Rightarrow u_x^2+v_x^2 = 0$ [$c^2+1 \neq 0$]

$\Rightarrow |f'(z)|^2 = 0$ [$f'(z) = u_x + iv_x$]

$\Rightarrow f'(z) = 0$

$\Rightarrow f(z)$ is constant $\forall z \in D$

(6) Let $f(z) = u+iv$ is an analytic function of z.

$u = \frac{\sin 2x}{\cosh 2y + \cos 2x}$, $\frac{\partial u}{\partial x} = \frac{2\cos 2x(\cosh 2y + \cos 2x) + \sin 2x \cdot 2\sin 2x}{(\cosh 2y + \cos 2x)^2}$

$$= \frac{2 + 2\cosh 2y \cos 2x}{(\cosh 2y + \cos 2x)^2} = \phi_1(x,y)$$

$$\frac{\partial u}{\partial y} = -\frac{\sin 2x \cdot 2\sinh 2y}{(\cosh 2y + \cos 2x)^2} = \phi_2(x,y)$$

$$\frac{\partial v}{\partial x} = \frac{2\sinh 2y \sin 2x}{(\cosh 2y + \cos 2x)^2} \qquad \left[\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right].$$

Therefore, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Int, $f(z) = \int [\phi_1(z,0) + i\,\phi_2(z,0)]\,dz + c$

$$= \int \frac{2 + 2\cos 2z}{(1+\cos 2z)^2}\,dz + c$$

$$= 2\int \frac{1}{1+\cos 2z}\,dz + c$$

$$= 2\int \frac{1}{2\cos^2 z}\,dz + c = \int \sec^2 z\,dz + c$$

$$= \tan z + c.$$

(c) $u = \frac{1}{2}\log(x^2+y^2)$,

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}. \qquad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad \therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$\Rightarrow u$ is harmonic function.

Let $f(z) = u + iv$ be analytic function of z.

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} = -\frac{\partial v}{\partial x} \qquad [u_y = -v_x]$$

$$\Rightarrow \frac{\partial v}{\partial x} = -\frac{y}{x^2+y^2}.$$

Int. w.r.to x we have, $v = -\int \frac{y}{x^2+y^2}\,dx + \phi(y)$

$$= -y \cdot \frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right) + \phi(y)$$

$$= -\tan^{-1}\left(\frac{x}{y}\right) + \phi(y).$$

$$\frac{\partial v}{\partial y} = -\frac{1}{1+\frac{x^2}{y^2}} \cdot \left(-\frac{x}{y^2}\right) + \phi'(y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} + \phi'(y) \Rightarrow \frac{x}{x^2+y^2} = \frac{x}{x^2+y^2} + \phi'(y) \Rightarrow \phi'(y) = 0.$$

$$\Rightarrow \phi(y) = c.$$

Therefore, $v = -\tan^{-1}\left(\frac{x}{y}\right) + c$.

(d) $f(z) = u + iv$ where $z = x + iy$, $\bar{z} = x - iy$

$\therefore x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$.

Then,
$$\frac{\partial f}{\partial \bar{z}} = \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right)$$
$$= \left(\frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2i}\frac{\partial u}{\partial y}\right) + i\left(\frac{1}{2}\frac{\partial v}{\partial x} - \frac{1}{2i}\frac{\partial v}{\partial y}\right)$$
$$= \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial u}{\partial y} + \frac{i}{2}\frac{\partial v}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y}$$
$$= \frac{1}{2}\frac{\partial v}{\partial y} - \frac{i}{2}\frac{\partial v}{\partial x} + \frac{i}{2}\frac{\partial v}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y} \quad \left[\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right]$$
$$= 0.$$

Cauchy-Riemann Equation in Polar form:

Prove that $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$ where $x = r\cos\theta$, $y = r\sin\theta$.

Proof: We have $x = r\cos\theta$, $y = r\sin\theta$ so $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$.

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \quad \frac{\partial r}{\partial y} = \sin\theta$$

Also, $$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right) = -\frac{\sin\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}.$$

Now, $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\cdot\frac{\sin\theta}{r}.$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\cdot\frac{\cos\theta}{r}$$

Similarly, $$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial \theta}\cdot\frac{\sin\theta}{r}$$
& $$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\cdot\frac{\cos\theta}{r}.$$

We have C-R equations $u_x = v_y$ and $u_y = -v_x$, ~~we have~~

$$\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\cdot\frac{\sin\theta}{r} = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\cdot\frac{\cos\theta}{r} \quad \text{--- (1)}$$

& $$\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\cdot\frac{\cos\theta}{r} = -\frac{\partial v}{\partial r}\cos\theta + \frac{\partial v}{\partial \theta}\cdot\frac{\sin\theta}{r} \quad \text{--- (2)}$$

Multiplying (1) by $\sin\theta$ and (2) by $\cos\theta$ and subtracting, $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

Again multiplying (1) by $\cos\theta$ and (2) by $\sin\theta$ and adding we have,
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

Thus, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$ are Cauchy's Riemann's equations in polar form.