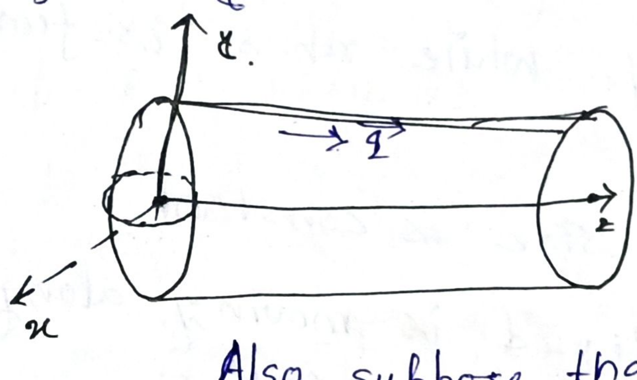
Steady flow through a cylindrical pipe :

Let an incompressible viscouss fluid ~~is~~ be in steady motion in a cylindrical pipe. And we take z axis along the axes of the cylinder.



Also suppose that the direction of flow is $\parallel^{le}$ to z axis. So that

$$\vec{q} = \vec{q}\,(0,0,w).$$

Then the equation of continuity,

$$\vec{\nabla}\cdot\vec{q} = 0, \quad \frac{\partial w}{\partial z} = 0 \quad \text{--------- (1)}$$

i.e, w is independent of z.

Now the Neivier - Stoke's eq^n ~~for~~ absence of body force for the steady state motion reduces to,

$$-\frac{1}{\rho}\vec{\nabla}p + \frac{\mu}{\rho}\nabla^2 w\hat{k} = \vec{0} \quad \text{--------- (2)}$$

(??)

Above eq^n can be writen as,

$$\underbrace{\frac{\partial p}{\partial x} = 0, \quad \frac{\partial p}{\partial y} = 0}_{(3.a)}, \quad \underbrace{\frac{\partial p}{\partial z} = \mu\nabla^2 w}_{(3.b)}$$

(3.a) & (3,b) shows that p is the fun^n of z only.

$\therefore$

Then (3·c) becomes,

$$\frac{dp}{dz} = \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \qquad (4)$$

From (4), we see that, l.h.s. is a funⁿ of z only, while r.h.s. is funⁿ of x, y only.

Hence each side is constant.

Since the liquid is moving along the positive direction of x axis,

So, p decreases as z increases

i.e, $\frac{dp}{dz} < 0$, $\forall z > 0$.

So, $\frac{dp}{dz} = -P$, say, for $P > 0$ (5)

From (5),

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{P}{\mu} \qquad (6)$$

Case I:

Above eqn in cylindrical co-ordinate can be written as

$$\frac{dp}{dz} = \mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)$$

$$= -P$$

, where $x = r\cos\theta$

$y = r\sin\theta$

$z = z$.

Since $w = w(x, y) = w(r)$.

So, we have ~~from~~ $\frac{dp}{dz} = \mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)$ -- (7)

and ~~from~~ $\frac{dp}{dz} = -P$

$\Rightarrow p = -Pz + A$, where A is constant. (8)

With the help of (8), the conditions,

i) $p = p_1$, when $z = z_1$

and ii) $p = p_2$, when $z = z_2$

gives,

$$p_1 = -Pz_1 + A$$

$$p_2 = -Pz_2 + A.$$

$$\therefore p_1 - p_2 = P(z_2 - z_1)$$

$$\Rightarrow P = \frac{p_1 - p_2}{l}, \text{ where } l = z_2 - z_1. \quad (9).$$

From (7), we have

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = -\frac{P}{\mu}.$$

Integrating,

$$r\frac{dw}{dr} = -\frac{Pr^2}{2\mu} + B.$$

Again integrating,

$$w = -\frac{Pr^2}{4\mu} + B\log r.$$

Since velocity w is finite along the axes, hence in perticulals at $r = 0$, we ~~have~~ make ~~B~~ $C =$

We must have $B = 0$

Hence $W = -\frac{Pr^2}{4\mu} + C$ ——— (11)

For no slip condition on the tube, we have $W = 0$, when $r = a$

So, from (11), we have

$$C = \frac{Pa^2}{4\mu}$$

Hence $W = \frac{P}{4\mu}(a^2 - r^2)$.

The rate of flow

$$Q = \int_0^a W \, 2r\pi \, dr$$

$$= 2\pi \int_0^a \frac{P}{4\mu}(a^2 - r^2) \, r \, dr$$

$$= \frac{P\pi}{2\mu} \int_0^a (ra^2 - r^3) \, dr$$

$$= \frac{P\pi}{2\mu}\left(\frac{a^4}{2} - \frac{a^4}{4}\right)$$

$$= \frac{p_1 - p_2}{L} \frac{\pi}{2\mu} \frac{a^4}{4}$$

$$= \frac{(p_1 - p_2)}{L} \frac{\pi a^4}{8}$$

Drag on the length l of the cylinder is given by

$$\text{Drag} = \left(\mu \frac{dw}{dr}\right)_{r=a} 2\pi a l$$

$$= -\pi a^2 (p_1 - p_2).$$

Maximum velocity occures on the axis of the cylinder at $r = 0$ and we have

$$W_{max} = \frac{P a^2}{4\mu} = \frac{a^2(p_1 - p_2)}{4\mu l}.$$

The result that flux (rate of flow) is proportional to the pressure gradiant and fourth power of radius, which was devoloped by G. Hagen and by Poiseuille. This result known as Hagen - Poiseuille flow.

$$\because Q = \frac{\pi a^4 (p_1 - p_2)}{l}$$

$$\Rightarrow Q \propto a^4$$

and $Q \propto \frac{p_1 - p_2}{l}.$

Case II : (Elliptic cross section)

If the cross-section of the pipe is elliptic section, then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

Then we have

$$W = W(x, y)$$

$$= K\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \text{ ——— } (2.7)$$

satisfy the condition of zero velocity on the walls of the pipe.

Then, for constant K, we have from (6),

$$K\left(\frac{-2K}{a^2}\right) \quad -\frac{2K}{a^2} - \frac{2K}{b^2} \quad K = \frac{P}{2\mu}\left(\frac{a^2}{a^2}\right)$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{P}{\mu}.$$

$$\Rightarrow K = \frac{P}{2\mu}\left(\frac{a^2 b^2}{a^2 + b^2}\right) \text{ ——— } (2.8)$$

(??)

$\therefore$ Total volume Q flowing per unit of time across the two sections is given by

$$Q = \iint w \, dx \, dy$$

$$= K \iint_{x\,y} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dx \, dy.$$

$$= K \pi ab \left[1 - \frac{a^2}{4}\frac{1}{a^2} - \frac{b^2}{4}\frac{1}{b^2}\right]$$

$$= \frac{K \pi ab}{2} \quad (3)$$

$\because \iint dx\,dy = \pi ab$

$\iint x^2 \, dx\,dy = \pi ab \frac{a^2}{4}$

$\iint y^2 \, dx\,dy = \pi ab \frac{b^2}{4}$

The ~~min~~ mean velocity $= \dfrac{Q}{\iint dx\,dy}$

$$= \frac{Q}{\pi ab} = \frac{K}{2}$$

Hence Flux $= Q = \dfrac{\pi}{4\mu} \cdot \dfrac{Pa^3b^3}{a^2+b^2}$

and corresponding min velocity $= \dfrac{K}{2}$

$$= \frac{P}{4\mu} \frac{a^2b}{a^2+b^2}$$

Case III : (Circular Cross section) :

Let the radius of the circular pipe be c, then we have

$$x^2+y^2 = c^2$$

So, by putting $a = b = c$ in case II,

we get

$$Q = \frac{\pi}{4\mu} \frac{Pc^6}{2c^2} = \frac{\pi Pc^4}{8\mu}$$

$$\text{Min velociy} = \frac{P}{4\mu} \frac{c^3}{2c^2} = \frac{Pc}{8\mu}$$

But, $\pi ab = \pi c^2$

i.e, $c^2 = ab$

$\Rightarrow c^2 = Ka^2$ for $b = aK$

So, we have $\dfrac{Q_{\text{ellipse}}}{Q_{\text{circle}}} = \dfrac{a^3b^3}{a^2+b^2}$

$$= \frac{2K}{1+K^2} < 1.$$

$$\Rightarrow \boxed{q_{ellipse} < q_{circle}}$$

$(k-1)^2 > 0$

$\Rightarrow 1 + k^2 > 2k$

$\Rightarrow \frac{2k}{1+k^2} < 1$

Boundary Layer theory :

Drag and Lift :

If a fluid flows in pressure of an ~~abstra~~ obstracle then the obstracle will experience two types of forces :

(i) One in the direction of motion of the fluid and

(ii) The other in a direction normal to the flow direction.

The first force is called drag force, we denote by D and $D = \left(\frac{1}{2} \rho q^2\right) A \times C_D$.

and 2nd force is known as lift force, denoted by L and $L = \left(\frac{1}{2} \rho q^2\right) A \times C_L$

, where A is area normal to flow direction, ρ = density

q = fluid velocity, C_D = Co-efficient of dra

C_L = Co-efficient of lift.

Actually these two forces are produce by tangential and normal stresses. The drag due to normal stress is called pressure drag and drag due to tangential stress. on shearing stresses ~~a~~ is called friction on skin friction.

Prandtl's boundary Layer theory:

The motion of the fluid due to rigid obstacle, by Prandtl's in 1904, divided into two domains

i) A thin domain very closed to the objects obstacle where viscous force (frictional force are prominant.

ii) A outer domain where the frictional force may be neglected.

In the outer domain, the fluid treated as non viscous.

- The first domain is known as boundary Layer.

Boundary layer theory is based upon the two assumptions:

(i) The effect of viscosity is appeneal only in a thin ~~flim~~ film arround the boundary of the rigid ~~bo~~ object, the rest of the fluid being treated as non viscous.

(ii) The pressure distribution is a non viscous fluid is known

Actually the flow within the boundary layer begins with laminar flow but as the layer grows along the surface a transition region occurs and the flow within the boundary layer may become turbulant if surface is very large.

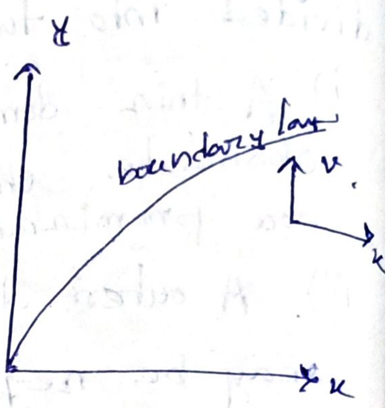
Boundary layer Equation :

let us consider two dimensional motion of incompressible fluid of small viscosity.

Taking x axis along the surface of wall and y axis normal to this surface

Then eqn of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{--- (1)}$$



and the N-S eqn of motion in absence of external forces are —

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \quad \text{--- (2)}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] \quad \text{--- (3)}$$

let δ be the thickness of the boundary layer and fluid velocity $\vec{q}(u,v)$ and the main stream with velocity $(U,0)$

Then u changes from 0 to U in the main stream in a length δ, i.e,

for $0 \le y \le \delta$, $u = 0 = v$ at $y = 0$ on velocity varises on the surface of solid and $u = 0$ at $y = \delta$.

Let the order of u be $O(u)$ and assume. $O(u) = O\left(\frac{\partial u}{\partial x}\right) = 1$ for $0 \le y \le \delta$

then $\frac{\partial u}{\partial y}$ is large as its order is $\frac{1}{\delta}$

and $O\left(\frac{\partial^2 u}{\partial y^2}\right) = \frac{1}{\delta^2}$

Thus from (1), we have

$$O\left(\frac{\partial v}{\partial y}\right) = 1 \quad \text{or} \quad O\left(\frac{\partial u}{\partial x}\right) = 1$$

and $v = 0$ then $y = 0 \Rightarrow O(v) = \delta$

$$O\left(\frac{\partial v}{\partial y}\right) = 1 \text{ and hence } O\left(\frac{\partial^2 v}{\partial y^2}\right) = \frac{1}{\delta}$$

$$\left[\because \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \frac{1}{\delta}\cdot 1 = \frac{1}{\delta}\right]$$

Hence (1) $u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}$ are all of order 1

(2) $v, \frac{\partial v}{\partial t}, \frac{\partial v}{\partial x}$ are all of order δ.

(3) ~~$v, \frac{\partial v}{\partial t}, \frac{\partial v}{\partial x}$~~ $O\left(\frac{\partial u}{\partial y}\right) = \frac{1}{\delta}$,

$O\left(\frac{\partial^2 u}{\partial y^2}\right) = \frac{1}{\delta^2}$, $O\left(\frac{\partial v}{\partial y}\right) = 1$, $O\left(\frac{\partial^2 v}{\partial y^2}\right) = \frac{1}{\delta}$

from (2) we have

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

$\left[\because \frac{\partial^2 u}{\partial x^2}\right.$ is negligible compare with $\frac{\partial^2 u}{\partial y^2}$

But the order of the viscous terms is the same as order of inertial term.

Hence $O\left(\frac{\nu}{\delta^2}\right) = 1$ as $O\left(\frac{\partial u}{\partial t}\right) = 1 = O\left(\nu\frac{\partial^2 u}{\partial y^2}\right)$

$$\Rightarrow O(\sqrt{\nu}) = O(\delta)$$

, this gives the estimate of the thickness of the boundary layer.

So, by eqⁿ ③ we see that every member of L.H.S. is order ~~set~~ of δ but R.H.S of ③ are

$$O\left(\nu \frac{\partial^2 v}{\partial x^2}\right) = O(\delta^2 \cdot \delta) = O(\delta^3)$$

and $$O\left(\nu \frac{\delta^2 v}{\delta y^2}\right) = O\left(\delta^2 \cdot \frac{1}{\delta}\right) = O(\delta)$$

Above result implies that all the terms in ③ are of order smaller than those in ②. Consequently pressure ~~from~~ term $\frac{\partial p}{\partial y}$ in ③ may be negligible as compared with $\frac{\partial p}{\partial x}$.

Hence ③ reduces to $\frac{\partial p}{\partial y} = 0$

$$\Rightarrow p = p(x) \text{ and } \frac{\partial p}{\partial x} = \frac{dp}{dx}.$$

Hence p = constant from $y = 0$ to $y = \delta$

Finally the diff: eqⁿ of the boundary layer are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{④}$$

and

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right) \qquad \text{⑤}.$$

This eqⁿ ④ and ⑤ are required eqⁿs of motion.

Deduction :

Bernoulli's eqn for steady motion is applicable in the main stream, i.e, in inviscous domain and we have

$$\frac{p}{\rho} + \frac{1}{2} q^2 = \text{Constant}$$

i.e, $\frac{1}{\rho}\frac{dp}{dx} + U\frac{\partial U}{\partial x} = 0$ $\left[\because q = U \text{ in inviscous domain outside the boundary}\right.$

Then (5) becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}.$$