Blasius Theorem :

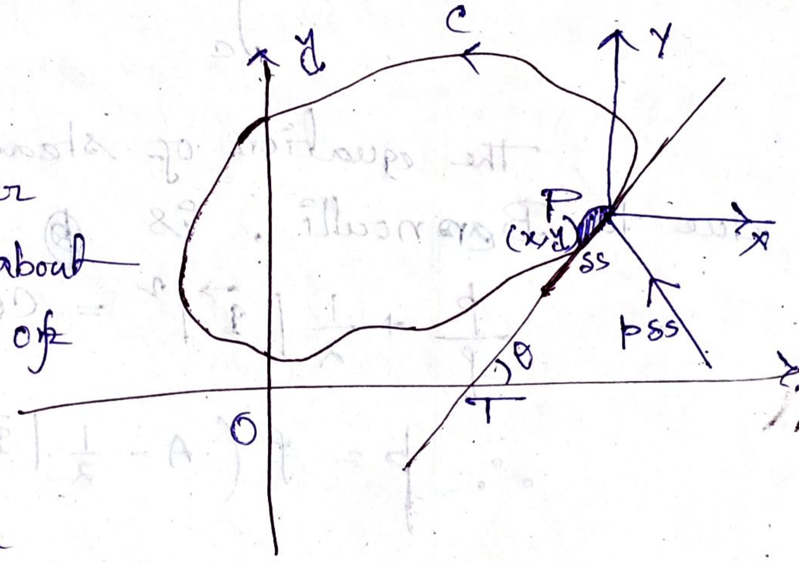
Statement : In steady two dimensional motion given by the complex potential $w = f(z) = \phi + i\psi$, if the pressure thrusts on the fixed cylinder of any shape are represented by a force (X, Y) and a couple of moment M about the origin of co-ordinates, then neglecting external forces and

$$X - iY = \frac{i\rho}{2} \int_C \left(\frac{dw}{dz}\right)^2 dz \quad \text{and}$$

$$M = Re\left[-\frac{\rho}{2} \int_C \left(\frac{dw}{dz}\right)^2 z\, dz\right]$$

, where ρ is the density and integrals are taken round the contour C of the cylinder.

Proof :



Let us consider an element δs about the point $P(x, y)$ of the fixed cylinder.

Let C be the boundary of the cylinder of any shape and size.

Let the tangent PT makes an angle θ with x axis.

$\therefore$ The inword normal at P makes an angle $\left(\frac{\pi}{2} + \theta\right)$ with x axis

Then the thrust components at P are

$p\delta s \cos(\pi/2 + \theta)$ along x axis and

$p\delta s \sin(\pi/2 + \theta)$ along y axis

i.e, $-p\delta s \sin\theta$ along x axis and

$p\delta s \cos\theta$ along y axis.

$$\therefore X = \sum -p\delta s \sin\theta = -\int_C p \sin\theta \, ds.$$

and

$$Y = \sum p \delta s \cos\theta = \int_C p \cos\theta \, ds.$$

$$\therefore X - iY = \int_C p(-\sin\theta - i\cos\theta)\, ds$$

$$= -i\int_C p(\cos\theta - i\sin\theta)\, ds$$

The equation of steady state mo[l]

due to Bernoulli, is

$$\frac{p}{\rho} + \frac{1}{2}|\vec{q}|^2 = \text{Constant} = A,$$

$$\therefore p = \rho\left(A - \frac{1}{2}|\vec{q}|^2\right)$$

Hence $X - iY = -i\rho \int_C \left(A - \frac{1}{2}|\vec{q}|^2\right)(c$

$$= \frac{1}{2} i\rho \int_C |\vec{q}|^2 e^{-i\theta} ds$$

$$- i\rho A \int_C e^{-i\theta} ds$$

2nd integral $= -i\rho A \int_C (\cos\theta \, ds - i \sin\theta \, ds)$

$$= -i\rho A \int_C (dx - i\,dy)$$

$\tan\theta = \frac{dy}{dx}$

$\sin\theta = \frac{dy}{ds}$

$\cos\theta = \frac{dx}{ds}$

$z = x + iy$

$$= -i\rho A \int_C d\bar{z}$$

$= 0$, by Cauch's theorem, as C closed contour.

Hence,

$$X - iY = \frac{1}{2} i\rho \int_C |\vec{q}|^2 e^{-i\theta} \, ds$$

Let u, v be the components of the veloc[ity] $\vec{q}$. i.e, $|\vec{q}|^2 = u^2 + v^2$.

and we know that

$$\frac{dw}{dz} = -u + iv$$

$\because u = -\frac{\partial \phi}{\partial x}$

$v = -\frac{\partial \phi}{\partial y}$

as $\vec{q} = -$

and c

$$= -|\vec{q}|\cos\theta + i|\vec{q}|\sin\theta$$

$$= -|\vec{q}|(\cos\theta - i\sin\theta)$$

$$= -|\vec{q}| e^{-i\theta}.$$

$$\left(\frac{dw}{dz}\right)^2 = |\vec{q}|^2 e^{-2i\theta}$$

$$X - iY = \left(\frac{dw}{dz}\right)^2 dz$$

$$= |\vec{q}|^2 e^{-2i\theta} (dx + i\,dy)$$

$$= |\vec{q}|^2 e^{-2i\theta}(\cos\theta\, ds + i\sin\theta\, ds)$$

$$= |\vec{q}|^2 e^{-2i\theta} e^{i\theta}\, ds$$

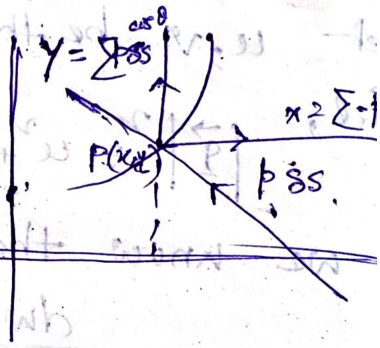
$$= |\vec{q}|^2 e^{-i\theta}\, ds$$

$$\therefore X - iY = \frac{1}{2} i\rho \int_C |\vec{q}|^2 e^{-i\theta}\, ds$$

$$= \frac{i\rho}{2}\int_C \left(\frac{dw}{dz}\right)^2 dz.$$

Which is the first result and moment of the thrust about the origin

Now the moment of thrust about origin O is.



$$N = \sum\left[(+p\,\delta s\cos\theta)\,x - y(-p\,\delta s\sin\theta)\right]$$

$$= \int_C (xp\cos\theta + yp\sin\theta)\, ds$$

$$= \int_C \rho\left(A - \frac{1}{2}|\vec{q}|^2\right)(x\,dx + y\,dy)$$

$$= \rho A \int_C \frac{d(x^2+y^2)}{2} - \frac{\rho}{2}\int_C |\vec{q}|^2 (x\,dx + y\,dy)$$

Now, $\int_C d(x^2+y^2) = 0$ (by Cauchy theorem.

Hence

$$N = -\frac{\rho}{2}\int_C |\vec{q}|^2 (x\,dx + y\,dy)$$

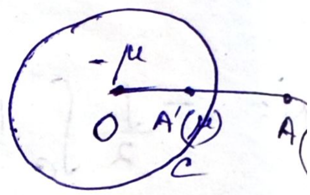
$$= Re\left[\frac{\rho}{2}\int_C |\vec{q}|^2\, z\, e^{-i\theta} ds\right]$$

$$\left[\begin{aligned} \because\ z e^{-i\theta} ds &= (x+iy)(\cos\theta\, ds - i\sin\theta\, ds) \\ &= (x+iy)(dx - i\,dy) \\ &= (x\,dx + y\,dy) + i(y\,dx - x\,dy) \\ \Rightarrow x\,dx + y\,dy &= Re\{z e^{-i\theta} ds\} \end{aligned}\right.$$

$$\therefore N = Re\left[-\frac{\rho}{2}\int_C z\left(\frac{dw}{dz}\right)^2 dz\right]$$

Hence the theorem.

Ex: A source of fluid situated in space of two dimension is of such strength that $2\pi\rho\mu$ represents that the mass of fluid of density ρ emitted for unit of time. Show that the force necessary to hold a circular disc at rest in the plane source is $\frac{2\pi\rho\mu^2 a^2}{r(r^2-a^2)}$, where a is the radius of the disc and r be the d of the source from its centre. In what direction is the disc urged by the pressure.



Sol^n: Let X and Y be the components of the required force. Then by Blasius theorem

$$X - iY = \frac{i\rho}{2}\int_C \left(\frac{dw}{dz}\right)^2 dz$$

, where C represents the boundary of the disc.

Since $2\pi\rho\mu$ represents the mass of the fluid emitted,

Hence straingth of the source is μ

Thus the image of a source $(+\mu)$ at A where $\{OA = r\}$, is a source $+\mu$ at the inverse point at A' s.t. $OA \cdot OA' = a^2$

and a sink $(-\mu)$ at O.

Now, $OA \cdot OA' = a^2$ gives $OA' = \frac{a^2}{r}$

$= r'$, say

$\therefore$ The complex potential due to object system with rigid boundary is equivalent to the complex potenti due to the object system and its image system with no rigid boundary.

Hence $W = -\mu \log(z-r) - \mu \log(z-r') + \mu$

$$\therefore \frac{dw}{dz} = -\frac{\mu}{z-r} - \frac{\mu}{z-r'} + \frac{\mu}{z}$$

$$= -\mu \left[\frac{1}{z-r} + \frac{1}{z-r'} - \frac{1}{z}\right]$$

or, $$\frac{1}{\mu^2}\left(\frac{dw}{dz}\right)^2 = \left[\frac{1}{z-r} + \frac{1}{z-r'} - \frac{1}{z}\right]^2$$

$$= \left[\frac{1}{(z-r)^2} + \frac{1}{(z-r')^2} + \frac{1}{z^2} + \frac{2}{(z-r}\right.$$

$$\left. - \frac{2}{z(z-r)} - \frac{2}{z(z-r')}\right]$$

From above we see that the function $f(z) = \frac{1}{\mu^2}\left(\frac{dw}{dz}\right)^2$ has poles at $z =$ and $z = r'$ within C but the pole at is outside of C.

Hence Res $(z=0)$ is the sum of the co of $\frac{1}{z}$ and hence ~~Res(z~~

$$\text{Res}(z=0) = \left[-\frac{2}{z-r'} - \frac{2}{z-r}\right]_{z=0}$$

$$= 2\left(\frac{1}{r} + \frac{1}{r'}\right)$$

Also Res $(z = r')$ = sum of the co-ef of $\frac{1}{z-r'}$ and hence

$$\text{Res}(z = r') = \left[\frac{2}{z-r} - \frac{2}{z}\right]_{z=r'}$$

$$= \frac{2}{r'-r} - \frac{2}{r'}$$

So sum of the Residues within C is

$$= 2\left(\frac{1}{r} + \frac{1}{r'}\right) + 2\left(\frac{1}{r'-r} - \frac{1}{r'}\right)$$

$$= 2\left(\frac{1}{r} + \frac{1}{r'-r}\right)$$

$$= \frac{2r'}{r(r'-r)}$$

$$= \frac{2a^2}{(a^2-r^2)r}$$

So, by Cauchy's Residue's theorem,

$$\int_C f(z)\,dz = \int_C \frac{1}{\mu^2}\left(\frac{dw}{dz}\right)^2 dz$$

$$= 2\pi i \times \text{sum of the residues within}$$

$$= \cancel{2\pi i \times} \frac{4\pi i a^2}{(a^2-r^2)r}$$

Also by Blasius theorem,

$$X - iY = \frac{i\rho}{2}\int_C \left(\frac{dw}{dz}\right)^2 dz$$

$$= \frac{i\rho}{2} \times \frac{4\pi a^2 i \mu^2}{(a^2-r^2)r}$$

Comparing with real and imaginary part,

$$X = -\frac{2\pi a^2 \mu^2 \rho}{(a^2-r^2)r} = \frac{2\pi a^2 \mu^2 \rho}{r(r^2-a^2)}$$

$$Y = 0$$

So, resultant force acting on the circle

is $\sqrt{X^2 + Y^2}$

$$= \frac{2\pi a^2 \mu^2 \rho}{r(r^2 - a^2)}$$

Above results shows that the force is purely along OA.

So the disc will be ~~urge~~ urged to move along OA.

Also the cylinder is attracted ~~towa~~ toward the source and sketch of the stream lines reveals that the pressure is great on the opposite side of the disc than that of the source.