Assignment

Paper- C13 :: Semt 6th :: Course- Mathematics (H) UG.

Subject- Complex Analysis

1. If $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$ find the corresponding analytic function $f(z) = u + iv$ by Milne's Thomson method.

2. Show that for the function $f = u + iv$, $\dfrac{\partial f}{\partial \bar{z}} = 0$

3. If $f(z)$ is an analytic function of z in any domain prove that
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^p = p^2 \, |f(z)|^{p-2} \, |f'(z)|^2$$

4. If $u - v = (x-y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$ find $f(z)$ in terms of z.

5. Evaluate $\int_C (x^2 - iy^2)\, dz$ along the parabola $y = 2x^2$ from $(1,1)$ to $(2,8)$.

6. Evaluate $\int_C (z^2 + 3z + 2)\, dz$ where C is the arc of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ between the points $(0,0)$ and $(\pi a, 2a)$.

7. Find the value of the integral $\int_0^{1+i} (x - y + ix)\, dz$ along the straight line from $z = 0$ to $z = 1+i$

8. Prove that if $f(z)$ is integrable along a curve C having finite Length L and if there exists a positive number M such that $|f(z)| \le M$ on C then
$$\left| \int_C f(z)\, dz \right| \le ML$$

9. Let $P(x,y)$ and $Q(x,y)$ be continuous and having continuous 1st partial derivatives at each point of a simply connected region R. Prove that a necessary and sufficient condition that $\oint_C P\,dx + Q\,dy = 0$ around every closed path C in R is that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ identically in R.

10. Prove Cauchy's Theorem $\oint_C f(z)\, dz = 0$ if $f(z)$ is analytic with derivative $f'(z)$ which is continuous at all points inside and on a simple closed curve C.

11. Prove that $\oint_C z\, dz = 0$ where C is any simple closed curve.

12. Evaluate $\oint \dfrac{dz}{z-2}$ around the circle $|z-2| = 5$

13. Prove the Cauchy Goursat theorem for any simple closed curve.

14. If $f(z)$ is analytic in simply connected region R prove that $\int_a^b f(z)\, dz$ is independent of the path in R joining any two points a and b in R

15. If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining the points $(1,1)$ and $(3,3)$ then find the value of $\int_C (12z^2 - 4iz)\, dz$.

16. State and prove Cauchy's integral formula.

17. If a function $f(z)$ is analytic in a region D, then its derivative at any point $z=a$ of D is also analytic in D and is given by
$$f'(a) = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)^2}$$ where c is any closed contour in D surrounding the point $z=a$.

18. If $f(z)$ is analytic within a circle C given by $|z-a|=R$ and if $|f(z)| \le M$ on C then $|f^n(a)| \le \frac{Mn!}{R^n}$.

19. State and prove Liouville's theorem.

20. If a function $f(z)$ is analytic for all finite values of z and as $|z| \to \infty$, $|f(z)| = A(|z|^k)$ then prove that $f(z)$ is a polynomial of degree $\le k$.

21. The function $f(z)$ is analytic when $|z|<R$ and has the Taylor's expansion $\sum_{n=0}^{\infty} a_n z^n$. Show that if $r<R$, $\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$.

Hence prove that if $|f(z)| \le M$ when $|z|<R$, $\sum_{n=0}^{\infty} |a_n|^2 r^{2n} < M^2$.

22. If C is the closed contour around origin, prove that
$$\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_c \frac{a^n e^{nz}}{n!\, z^{n+1}}\,dz.$$ Hence deduce $\sum_{n=0}^{\infty}\left(\frac{a^n}{\lfloor n}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2a\cos\theta}\,d\theta$.

23. If $f(z)$ is analytic in a region R prove that $f'(z), f''(z), \ldots$ are analytic in R.

24. Evaluate (i) $\oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$ (ii) $\oint_c \frac{e^{2z}}{(z+1)^4}\,dz$ where c is the circle $|z|=3$.

25. State and prove fundamental theorem of algebra.

26. Prove that every polynomial equation $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$ $(a_n \ne 0)$ and $n \ge 1$, has exactly n roots.

27. Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3 4^n}$.

28. Test the uniform convergence of $\sum_{n=1}^{\infty} \frac{z^n}{n\sqrt{n+1}}$, $|z| \le 1$.

29. Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent series valid for (i) $1<|z|<3$ (ii) $|z|>3$ (iii) $0<|z+1|<2$ (d) $|z|<1$.

30. For the function $f(z) = \frac{2z^3+1}{z^2+z}$ (a) find a Taylor's series valid in the nbd of the point $z=i$ (b) a Laurent's series valid within the annulus of which centre is origin.