MATRIX

properties of symmetric and Skew - symmetric matrices.

($a_{ij} = a_{ji}$
$a_{ij} = -a_{ji}$)

1. The product of a matrix and its transpose is a symmetric matrix.

[Let, A be an $m \times n$ matrix so that A^T is an $n \times m$ matrix. Then AA^T is a square matrix of order m.

Now, we have, $[AA^T] = [A^T]^T A^T$

$= AA^T$

So, AA^T is symmetric.

2. The product of two symmetric matrices of the same order is symmetric provided the product is commutative.

Let, A and B ~~as~~ be two symmetric matrices of the same order. Here $AB = BA$

so, we have

$$[AB]^T = B^T A^T = BA = AB$$

So, AB is Symmetric.

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2	3	4	5	6	7	15
9	10	11	12	13	14	22
16	17	18	19	20	21	29
23	24	25	26	27	28	

3) If A be a Square matrix, then $(A+A^T)$ is Symmetric and $(A-A^T)$ is Skew-symmetric.

$$(A+A^T)^T = A^T + (A^T)^T = A^T + A$$

So, $(A+A^T)$ is Symmetric

$$(A-A^T)^T = A^T - A = -(A-A^T)$$, Hence $(A-A^T)$ is skew-symmetric

4) Any Square matrix can be expressed uniquely as the sum of two matrices. one symmetric and the other Skew-Symmetric

properties of matrix addition

(i) Matrix addition is Commutative: $A+B = B+A$

(ii) " " " Associative: $A+(B+C) = (A+$

Transpose of A is A' or A^T

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Conjugate of a matrix $= \bar{A}$

Conjugate transpose of a matrix: A^θ or $\bar{A}'$ or $\bar{A}^T$

If $A = \begin{bmatrix} 2+i & 3-4i \\ 4i & 6 \end{bmatrix}$ find $A', \bar{A}, \bar{A}', A^\theta, (A^\theta)'$

$\Rightarrow$

$A' = \begin{bmatrix} 2+i & 4i \\ 3-4i & 6 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 2-i & 3+4i \\ -4i & 6 \end{bmatrix}$

$\bar{A}' = \begin{bmatrix} 2-i & -4i \\ 3+4i & 6 \end{bmatrix}; \quad A^\theta = \bar{A}' = \begin{bmatrix} 2-i & -4i \\ 3+4i & 6 \end{bmatrix}$

prove that $A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$ is a unitary, where ω is complex cube root of unity

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$\Rightarrow \quad A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$

$A' = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}; \quad A^\theta = \bar{A}' = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$

$1+\omega+\omega^2 = 0$

$\therefore A^\theta A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$

$= \frac{1}{3} \begin{bmatrix} 3 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ 1+\omega+\omega^2 & 1+2\omega^3 & 1+\omega^2+\omega^4 \\ 1+\omega+\omega^2 & 1+\omega^2+\omega^4 & 1+2\omega^3 \end{bmatrix}$

$= \frac{1}{3} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1+\omega^2(1+\omega^2) \\ 0 & 1+\omega^2(1+\omega^2) & 3 \end{bmatrix}$

$1+w^2+w=0$; $1+w^2=w$, $\omega^3=1$

$$A^{\theta}.A = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence the given matrix is unitary

Find the eigen values and eigen vectors of the matrix

$$\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$$

$\Rightarrow$ $A-\lambda I = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{bmatrix}$

The characteristic equ.^{n} is $|A-\lambda I| = 0$

$\lambda^2 - 5\lambda - 6 = 0$

$\lambda^2 - 6\lambda + \lambda - 6 = 0$

$\lambda+1 = 0$ / $\lambda - 6 = 0$

$\lambda = -1$, $\lambda = 6$ So, the eigen values are 6, -1

$\underline{\lambda = 6}$,

$$\begin{bmatrix} 1-6 & -2 \\ -5 & 4-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-5x_1 - 2x_2 = 0$

$\frac{x_1}{2} = \frac{x_2}{-5}$ Giving eigen vector $(2, -5)$

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But for, $\lambda = -1$ we have

$$\begin{bmatrix} 1-6(-1) & -2 \\ -5 & 4-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x_1 - 2x_2 = 0$ or, $x_1 = x_2$

i.e $\frac{x_1}{1} = \frac{x_2}{1}$, so that eigen vectors are $(1,1)$

$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} = ??$ $\quad (2-\lambda)^3 - (2-\lambda) = 0$

$(2-\lambda)(\lambda-1)(\lambda-3) = 0$

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verify Cayley Hamilton Theorem for the matrix

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

$\Rightarrow$ The characteristic equ.^{n} of the matrix

$$|A - \lambda I| = 0 \; ; \; \lambda^2 - 7\lambda + 1 = 0$$

so to verify Cayley Hamilton Theorem, we have to show that

$$A^2 = A \cdot A = \begin{bmatrix} 34 & 21 \\ 21 & 13 \end{bmatrix} ; \; 7A = \begin{bmatrix} 35 & 21 \\ 21 & 14 \end{bmatrix}$$

$$A^2 - 7A + I = \begin{bmatrix} 34 & 21 \\ 21 & 13 \end{bmatrix} - \begin{bmatrix} 35 & 21 \\ 21 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

so, theorem is verified.

verify Cayley Hamilton Theorem for the matrix $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and find its inverse

$\Rightarrow \quad A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$|A-\lambda I| = \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix} = 0$

$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

$A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$

$A^3 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$

i.e $A^3 - 6A^2 + 9A - 4I = 0$

i.e Caley Hamilton theorem is verified

$A^2 - 6A + 9I - 4IA^{-1} = 0$

$A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$

$= \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

Cayley Hamilton Theorem :-

Every square matrix satisfy its own characteristic equation. So if A is a square matrix of order n then the characteristic polynomial for nth order square matrix A is

$$|A-\lambda I| = (-1)^n (\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n)$$

then matrix eqn

$$X^n + a_1X^{n-1} + a_2X^{n-2} + \cdots + a_nI = 0$$

is satisfied by $X = A$, i.e $A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_nI = 0$

Proof :- Let the adjoint of the matrix $A-\lambda I$ be P.

Then elements of P will be polynomial of $(n-1)^{th}$ degree in λ, so that it consists of a number of matrice, such that $P = P_0\lambda^{n-1} + P_1\lambda^{n-2} + \cdots + P_{n-1}$

where, $P_0, P_1, \ldots, P_{n-1}$ are all the square matrices of order n whose elements are function of elements of A. But it is known that $|A-\lambda I| P = |A-\lambda I| I$

or, $(A-\lambda I)(P_0\lambda^{n-1} + P_1\lambda^{n-2} + \cdots + P_{n-1})$

$$= (-1)^n (\lambda^n + a_1\lambda^{n-1} + \cdots + a_n) I$$

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Now equating coefficients of like power of λ on both sides of equ.^{n} (iii), we get.

$$-IP_0 = (-1)^n I$$

$$AP_0 - IP_1 = (-1)^n a_1 I$$
$$AP_1 - IP_2 = (-1)^n a_2 I$$

— — — — — — — —

$$AP_{n-1} = (-1)^n a_n I$$

Hence by multiplying the equations by A^0, A^{n-1} ... I respectively and adding we get

$$0 = (-1)^n [A^n + a_1 A^{n-1} + \cdots + a_n I]$$

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i,e $\quad A^n + a_1 A^{n-1} + \cdots + a_n I = 0$

which is prove the theorem

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DIAGONALISATION OF A MATRIX

Diagonalise the following matrix $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$\Rightarrow$ The characteristic equ^n is $|A-\lambda I| = 0$

$$\Rightarrow (\lambda-1)(\lambda-3) = 0 \;/\; \lambda = -1, 3$$

which are the characteristics roots or eigen values

Hence the Diagonal matrix is given by

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

Diagonalise of the matrix $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

$$|A-\lambda I| = -\lambda^3 + \lambda^2 + 5\lambda - 5 = 0$$

$\therefore$ Eigen values are $1, \sqrt{5}, -\sqrt{5}$

So, the diagonal matrix is given by

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$

* Diagonalisation of a matrix is the process of reduction of a square matrix to diagonal form. If P is a non singular matrix then $D = P^{-1}AP$ is called Diagonal Matrix. The elements of the diagonal matrix are the eigenvalues.

$A \to$ modal matrix

Prob

Coupled linear ordinary differential equⁿ

When two or more dependent variable are functions of single independent variable, the equations involving their derivatives are called coupled differential equ.ⁿ

① **Ex** – $dx/dt = -2x+y$; $dy/dt = x-2y$

$\Rightarrow x_1 = x, \; x_2 = y$

$dx_1/dt = -2x_1 + x_2$; x_1 and x_2 are function of 't'

$\dot{x}_1 = -2x_1 + x_2$

similarly, $\frac{dx_2}{dt} = x_1 - 2x_2$

$\dot{x}_2 = x_1 - 2x_2$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

let, $x_1 = Ae^{\lambda t}$, $\lambda_2 = Ae^{\lambda t}$

The eigen value problem is

$$\left| \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2+\lambda)^2 = 1$; $\lambda = -1; -3$

for, $\lambda = -1$

we get
$$\begin{bmatrix} -2-(-1) & 1 \\ 1 & -2-(-1) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

or,
$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

i,e $A_1 = A_2$ eigen vector are $(1,1)$

$\lambda = -3$
$$\begin{bmatrix} -2-(-3) & 1 \\ 1 & -2-(-3) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

$A_1 + A_2 = 0$; $A_1 = -A_2$ eigen

vectors are $(1,-1)$

$\therefore$ The general sol.^{n} is

$$X = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$$

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EX-2 $\frac{dx_1}{dt} = x_1 + x$; $dx_2/dt = 4x_1 + x$

$\Rightarrow$ In matrix form $\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

let, sol^n be $(x_1, x_2) = (A_1 A_1) e^{\lambda t}$

Eigen value problem is

$$\left[\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = 0$$

or $1-\lambda = \pm 2$ or, $\lambda = 1 \pm 2 = 3, -1$

For $\lambda = 3$; $\begin{bmatrix} 1-3 & 1 \\ 4 & 1-3 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} -2A_1 + A_2 \\ 4A_1 - 2A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A_2 = 2A_1$$

$$\frac{A_1}{1} = \frac{A_2}{2}$$

$\therefore$ Eigen vectors are $(1, 2) \Rightarrow \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For $\lambda = -1$; $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow 2A_1 + A_2 = 0$

$4A_1 + 2A_2 = 0$

i,e $A_1 = -\frac{A_2}{2}$ i,e $\frac{A_1}{1} = \frac{A_2}{-2}$

$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\therefore$ General sol.^{n} is $X = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} e$

H.W $X' = \begin{bmatrix} 2 & 7 \\ -5 & 10 \end{bmatrix} X$ where $X' = \frac{dx}{dt}$

Ans. $X = c_1 \begin{bmatrix} 7 \\ 5 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-5t}$

H.W solve $X' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} X$, where

$X' = \frac{dx}{dt}$

Ans $= c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}$

Follow → H.K Das

→ B.S. Grewal

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